Formal Specification—
Z Notation—
Syntax, Type and Semantics

Consensus Working Draft 2.6

August 24, 2000

Developed by members of the Z Standards Panel

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ISO Panel JTC1/SC22/WG19 (Rapporteur Group for Z)
Project number JTC1.22.45

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This is a Working Draft by the Z Standards Panel. It has evolved from Consensus Working Draft 2.5 of June 20, 2000 according to remarks received and discussed at Meeting 56 of the Z Panel.

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Foreword

ISO (the International Organization for Standardization) and IEC (the International Electrotechnical Commission) form the specialized system for worldwide standardization. National bodies that are members of ISO or IEC participate in the development of International Standards through technical committees established by the respective organization to deal with particular fields of technical activity. ISO and IEC technical committees collaborate in fields of mutual interest. Other international organizations, governmental and non-governmental, in liaison with ISO and IEC, also take part in the work.

International Standards are drafted in accordance with the rules given in the ISO/IEC Directives, Part 3.

In the field of Information Technology, ISO and IEC have established a joint technical committee, ISO/IEC JTC 1. Draft International Standards adopted by the joint technical committee are circulated to national bodies for voting. Publication as an International Standard requires approval by at least 75% of the national bodies casting a vote.

Attention is drawn to the possibility that some of the elements of this International Standard may be the subject of patent rights. ISO and IEC shall not be held responsible for identifying any or all of such patent rights.

International Standard ISO/IEC 13568 was prepared by Joint Technical Committee ISO/IEC JTC 1, *Information Technology*, Subcommittee SC 22, *Programming languages, their environments and system software interfaces.*

Annexes A to C form a normative part of this International Standard. Annexes D and E are for information only.

Introduction

This International Standard specifies the syntax, type and semantics of the Z notation, as used in formal specification.

A specification of a system should aid understanding of that system, assisting development and maintenance of the system. Specifications need express only abstract properties, unlike implementations such as detailed algorithms, physical circuits, etc. Specifications may be loose, allowing refinement to many different implementations. Such abstract and loose specifications can be written in Z.

A specification written in Z models the specified system: it names the components of the system and expresses the constraints between those components. The meaning of a Z specification—its semantics—is defined as the set of interpretations (values for the named components) that are consistent with the constraints.

Z uses mathematical notation, hence specifications written in Z are said to be formal: the meaning is captured by the form of the mathematics used, independent of the names chosen. This formal basis enables mathematical reasoning, and hence proofs that desired properties are consequences of the specification. The soundness of inference rules used in such reasoning should be proven relative to the semantics of the Z notation.

This International Standard establishes precise syntax and semantics for a system of notation for mathematics, providing a basis on which further mathematics can be formalized.

Particular characteristics of Z include:

- its extensible toolkit of mathematical notation;
- its schema notation for specifying structures in the system and for structuring the specification itself; and
- its decidable type system, which allows some well-formedness checks on a specification to be performed automatically.

Examples of the kinds of systems that have been specified in Z include:

- safety critical systems, such as railway signalling, medical devices, and nuclear power systems;
- security systems, such as transaction processing systems, and communications; and
- general systems, such as programming languages and floating point processors.

Standard Z will also be appropriate for use in:

- formalizing the semantics of other notations, especially in standards documents.

This is the first ISO standard for the Z notation. Much has already been published about Z. Most uses of the Z notation have been based on the examples in the book "Specification Case Studies" edited by Hayes [2][3]. Early definitions of the notation were made by Sufrin [13] and by King *et al* [7]. Spivey's doctoral thesis showed that the semantics of the notation could be defined in terms of sets of models in ZF set theory [10]. His book "The Z Notation—A Reference Manual" [11][12] is the most complete definition of the notation, prior to this International Standard. Differences between Z as defined here and as defined in [12] are discussed in [14]. This International Standard addresses issues that have been resolved in different ways by different users, and hence encourages interchange of specifications between diverse tools. It also aims to be a complete formal definition of Z.

Formal Specification—
Z Notation—
Syntax, Type and Semantics

1 Scope

The following are within the scope of this International Standard:

- the syntax of the Z notation;
- the type system of the Z notation;
- the semantics of the Z notation;
- a toolkit of widely used mathematical operators;
- LaTeX [9] and email mark-ups of the Z notation.

The following are outside the scope of this International Standard:

- any method of using Z, though an informative annex (E) describes one widely-used convention.

2 Normative references

The following normative documents contain provisions which, through reference in this text, constitute provisions of this International Standard. For dated references, subsequent amendments to, or revisions of, any of these publications do not apply. However, parties to agreements based on this International Standard are encouraged to investigate the possibility of applying the most recent editions of the normative documents indicated below. For undated references, the latest edition of the normative document referred to applies. Members of ISO and IEC maintain registers of currently valid International Standards.

ISO/IEC 10646-1:2000, *Information Technology—Universal Multiple-Octet Coded Character Set (UCS)—Part 1: Architecture and Basic Multilingual Plane, plus its amendments and corrigenda*

ISO/IEC FCD 10646-2, *Information Technology—Universal Multiple-Octet Coded Character Set (UCS)—Part 2: Secondary Multilingual Plane for scripts and symbols, Supplementary Plane for CJK Ideographs, Special Purpose Plane*

ISO/IEC 14977:1996, *Information Technology—Syntactic Metalanguage—Extended BNF*

3 Terms and definitions

For the purposes of this International Standard, the following definitions of terms apply. *Italicized* terms in definitions are themselves defined in this list.

3.1
binding
finite function from names to values

3.2
capture
cause a reference expression to refer to a different declaration from that intended

3.3
carrier set
set of all values in a type

3.4
constraint
property that is either true or false

3.5
environment
function from names to information used in type inference

3.6
interpretation
function from global names of a section to values in the *semantic universe*

3.7
metalanguage
language used for defining another language

3.8
metavariable
name denoting an arbitrary phrase of a particular syntactic class

3.9
model
interpretation that makes the defining *constraints* of the corresponding section be true

3.10
schema
set of *bindings*

3.11
scope of a declaration
part of a specification in which a reference expression whose name is the same as a particular declaration refers to that declaration

3.12
scope rules
rules determining the *scope of a declaration*

3.13
semantic universe
set of all semantic values, providing representations for both non-generic and generic Z values

3.14
signature
function from names to types

3.15
type universe
set of all type values, providing representations for all Z types

3.16
ZF set theory
Zermelo-Fraenkel set theory

4 Symbols and definitions

For the purposes of this International Standard, the following definitions of symbols apply.

4.1 Syntactic metalanguage

The syntactic metalanguage used is the subset of the standard ISO/IEC 14977:1996 [6] summarised in Table 1, with modifications so that the mathematical symbols of Z can be presented in a more comprehensible way.

Table 1 – Syntactic metalanguage

Symbol	Definition
`=`	defines a non-terminal on the left in terms of the syntax on the right.
`|`	separates alternatives.
`,`	separates notations to be concatenated.
—	separates notation on the left from notation to be excepted on the right.
`{ }`	delimit notation to be repeated zero or more times.
`[ ]`	delimit optional notation.
`( )`	are grouping brackets (parentheses).
`' '`	delimit a terminal symbol.
`;`	terminates a definition.
(*)*	delimit commentary.

The infix operators `|` and `,` have precedence such that parentheses are needed when concatenating alternations, but not when alternating between concatenations. The exception notation is always used with parentheses, making its precedence irrelevant. Whitespace separates tokens of the syntactic metalanguage; it is otherwise ignored.

EXAMPLE The lexis of a `NUMERAL` token, and its informal reading, are as follows.

```
NUMERAL            = NUMERAL , DIGIT
                   | DIGIT
                   ;
```

The non-terminal symbol `NUMERAL` stands for a maximal sequence of one or more digit characters (without intervening white space).

The changes to ISO/IEC 14977:1996 allow use of mathematical symbols in the names of non-terminals, and are formally defined as follows.

NOTE 1 The question marks delimit the formal definition, as required by ISO/IEC 14977:1996.

```
?
Meta identifier character  =  all cases from ISO/IEC 14977:1996
                           |  '⊢' | '∀' | '∃' | '•' | '⇔' | '⇒' | '∨' | '∧' | '¬' | '∈'
                           |  'λ' | 'μ' | '⨟' | '≫' | '↾' | '×' | 'ℙ' | 'θ'
                           |  '{' | '}' | '(' | ')' | '[' | ']' | '⦉' | '⦊' | '⟪' | '⟫'
                           |  '|' | ',' | '.' | '/' | ';' | ':' | '=' | '\' | '_' | '&' | '⨾'
                           ;
?
```

NOTE 2 This anticipates a future version of ISO/IEC 14977:1996 permitting use of the characters of ISO/IEC 10646. It also anticipates a future version of ISO/IEC 10646 including all of these mathematical symbols (most are already included and the rest have been proposed).

The new `Meta identifier characters` '(', ')', '[', ']', '{', '}', ',', '|', '&', '; ' and '=' overload existing metalanguage characters. Uses of them as `Meta identifier characters` are with the common suffix -tok, e.g. (-tok, which may be viewed as a postfix metalanguage operator.

A further change to ISO/IEC 14977:1996 is the use of multiple fonts: metalanguage characters and non-terminals are in `Typewriter`, those non-terminals that correspond to Z tokens appear as those Z tokens normally appear, typically in Roman, and comments are in *Italic*.

The syntactic metalanguage is used in defining Z characters, lexis, concrete syntax and annotated syntax.

4.2 Mathematical metalanguage

4.2.1 Introduction

Logic and Zermelo-Fraenkel set theory are the basis for the semantics of Z. In this section the specific notations used are described. The notations used here are deliberately similar in appearance to those of Z itself, but are grounded only on the logic and set theory developed by the wider mathematical community.

The mathematical metalanguage is used in type inference rules and in semantic relations.

4.2.2 Parentheses

The forms of proposition and expression are given below. Where there could be any ambiguity in the parsing, usually parentheses have been used to clarify, but in any other case the precedence conventions of Z itself are intended to be used.

The use of parentheses is given in tabular form in Table 2, where p stands for any proposition and e stands for any expression.

Table 2 – Parentheses in metalanguage

Notation	Definition
(p)	p
(e)	e

The same brackets symbols are used around pairs and tuple extensions (Table 12), but those cannot be omitted in this way.

4.2.3 Propositions

4.2.3.1 Introduction

The value of a metalanguage proposition is either *true* or *false*. The values *true* and *false* are distinct. In this International Standard, no proposition of the metalanguage is both *true* and *false*; that is, this metatheory is consistent. Furthermore, every proposition is either *true* or *false*, even where it is not possible to say which; that is, the logic is two-valued.

4.2.3.2 Propositional connectives

The propositional connectives of negation, conjunction and disjunction are used. In Table 3 and later, p, p_2, etc, represent arbitrary propositions.

Table 3 – Propositional connectives in metalanguage

Notation	Name	Definition
$\neg\, p$	negation	*true* iff p is *false*
$p_1 \land p_2$	conjunction	*true* iff p_1 and p_2 are both *true*
$p_1 \lor p_2$	disjunction	*false* iff p_1 and p_2 are both *false*

Conjunction is also sometimes indicated by writing propositions on successive lines, as a vertical list.

4.2.3.3 Quantifiers

Existential, universal and unique-existential quantifiers are used. In Tables 4 and 5 and later, i, i_2, etc, represent arbitrary names, e, e_2, etc, represent arbitrary expressions, and ... represents zero or more repetitions of the surrounding formulae; in these tables, the propositions can contain references to the names, but the expressions cannot.

Table 4 – Quantifiers in metalanguage

Notation	Name	Definition
$\exists\ i_1 : e_1;\ ...;\ i_n : e_n \bullet p$	existential quantification	there exist ($\exists$) values of i_1 in set e_1, ..., i_n in set e_n ($i_1 : e_1;\ ...;\ i_n : e_n$) such that ($\bullet$) p is *true*
$\forall\ i_1 : e_1;\ ...;\ i_n : e_n \bullet p$	universal quantification	for all ($\forall$) values of i_1 in set e_1, ..., i_n in set e_n ($i_1 : e_1;\ ...;\ i_n : e_n$), it is true that ($\bullet$) p is *true*
$\exists_1\ i_1 : e_1;\ ...;\ i_n : e_n \bullet p$	unique existential quantification	there exists exactly one ($\exists_1$) configuration of values i_1 in set e_1, ..., i_n in set e_n ($i_1 : e_1;\ ...;\ i_n : e_n$) such that ($\bullet$) p is *true*

Certain abbreviations in the writing of quantifications are permitted, as given in Table 5. They shall be applied repeatedly until none of them is applicable.

Table 5 – Abbreviations in quantifications in metalanguage

Notation	Definition
$i_1, i_2, ..., i_n : e$	$i_1 : e;\ i_2, ..., i_n : e$
$\exists\ i_1 : e_1;\ ...;\ i_n : e_n \mid p_1 \bullet p_2$	$\exists\ i_1 : r_1;\ ...;\ i_n : e_n \bullet p_1 \wedge p_2$
$\forall\ i_1 : e_1;\ ...;\ i_n : e_n \mid p_1 \bullet p_2$	$\forall\ i_1 : e_1;\ ...;\ i_n : e_n \bullet (\neg\ p_1) \vee p_2$
$\exists_1\ i_1 : e_1;\ ...;\ i_n : e_n \mid p_1 \bullet p_2$	$\exists_1\ i_1 : e_1;\ ...;\ i_n : e_n \bullet p_1 \wedge p_2$

4.2.3.4 Conditional expression

The conditional expression allows the choice between two alternative values according to the truth or falsity of a given proposition, as defined in Table 6.

Table 6 – Conditional expression in metalanguage

Notation	Definition
if p then e_1 else e_2	either p is *true* and e_1 is the value, or p is *false* and e_2 is the value

4.2.4 Sets

4.2.4.1 Introduction

The notation used here is based on Zermelo-Fraenkel set theory, as described in for example [1], and the presentation here is guided by the order given there. In that theory there are only sets. Members of sets can only be other sets. The word "element" may be used loosely when referring to set members treated as atomic, without regard to their set nature. If metalanguage operations are applied to inappropriate arguments, they produce unspecified results rather than being undefined.

4.2.4.2 The universe

The universe, $\mathbb{W}$, denotes a world of sets, providing semantic values for Z expressions. $\mathbb{W}$ is big enough to contain the set `NAME`, from which Z names are drawn, and an infinite set and be closed under formation of powerset and products. The formation of a suitable $\mathbb{W}$ comprising models of sets, tuples and bindings, as needed to model Z, is well-known in ZF set theory and is assumed in this International Standard.

4.2.4.3 Propositions about sets and elements

The simplest propositions about sets are the relationships of membership, non-membership, subset and equality between sets or their elements, as detailed in Table 7.

Table 7 – Propositions about sets in metalanguage

Notation	Name	Definition
$e_1 \in e_1$	membership	*true* iff e_1 is a member of set e_2
$e_1 \notin e_2$	non-membership	$\neg\ e_1 \in e_2$
$e_1 \subseteq e_2$	subset	$\forall\ i : e_1 \bullet i \in e_2$
$e_1 = e_2$	equality	for e_1 and e_2 considered as sets, $e_1 \subseteq e_2 \wedge e_2 \subseteq e_1$

4.2.4.4 Basic set operations

ZF set theory constructs its repertoire of set operations starting with the axiom of empty set, then showing how to build up sets using the axioms of pairing and of union, and how to trim them back with the axiom of subset or separation.

For the purposes of this mathematical metalanguage, the simplest form of set comprehension is defined directly using the axiom of separation in Table 8. The existence of a universal set $\mathbb{W}$ is assumed. Other forms of set comprehension are defined in terms of the simplest form, using rules in which w is any name distinct from those already in use. Also defined in Table 8 are notations for empty set, finite set extensions, unions, intersections and differences.

Table 8 – Basic set operations in metalanguage

Notation	Name	Definition
$\{i : e \mid p\}$	set comprehension	subset of elements i of e such that p, by axiom of separation
$\{i_1 : e_1;\ ...;\ i_n : e_n \bullet e\}$	set comprehension	$\{w : \mathbb{W} \mid \exists\ i_1 : e_1;\ ...;\ i_n : e_n \bullet w = e\}$
$\{i_1 : e_1;\ ...;\ i_n : e_n \mid p \bullet e\}$	set comprehension	$\{w : \mathbb{W} \mid \exists\ i_1 : e_1;\ ...;\ i_n : e_n \bullet p \wedge w = e\}$
$\varnothing$	empty set	$\{i : \mathbb{W} \mid \mathit{false}\}$
$\{\}$	empty set	$\{i : \mathbb{W} \mid \mathit{false}\}$
$\{e\}$	singleton set	$\{i : \mathbb{W} \mid i = e\}$
$e_1 \cup e_2$	union	$\{i : \mathbb{W} \mid i \in e_1 \vee i \in e_2\}$
$\{e_1, e_2, ..., e_n\}$	set extension	$\{e_1\} \cup \{e_2, ..., e_n\}$
$e_1 \cap e_2$	intersection	$\{i : e_1 \mid i \in e_2\}$
$e_1 \setminus e_2$	difference	$\{i : e_1 \mid i \notin e_2\}$

4.2.4.5 Powersets

The axiom of powers asserts the existence of a powerset, which is the set of all subsets of a set. The set of all finite subsets is a subset of the powerset. It is the smallest set containing the empty set and all singleton subsets of e and closed under the operation of forming the union with singleton subsets of e. Their forms are given in Table 9.

4.2.5 Numbers

Numbers are not primitive in Zermelo-Fraenkel set theory, but there are several well established ways of representing them. The choice of coding is irrelevant here and so is not specified. There are notations to measure the cardinality of a finite set, to define addition of natural numbers and to form the set of natural numbers between two stated natural numbers, as given in Table 10.

Table 9 – Powerset in metalanguage

Notation	Name	Definition
$\mathbb{P}\ e$	set of all subsets	$\{i : \mathbb{W} \mid i \subseteq e\}$
$\mathbb{F}\ e$	set of all finite subsets	$\{i_1 : \mathbb{W} \mid \forall\, i_2 : \mathbb{P}\,\mathbb{P}\, e \mid \varnothing \in i_2 \land (\forall\, i_3 : i_2 \bullet \forall\, i_4 : e \bullet i_3 \cup \{i_4\} \in i_2) \bullet i_1 \in i_2\}$

Table 10 – Operations on numbers in metalanguage

Notation	Definition
$e_1 + e_2$	sum of natural numbers e_1 and e_2
$\#\, e$	cardinality of finite set e
$e_1 \mathrel{..} e_2$	set of natural numbers between e_1 and e_2 inclusive

4.2.6 Names

Names are needed for this International Standard. There are several ways of representing names in Zermelo-Fraenkel set theory. The choice of coding is irrelevant here and so is not specified. Only one operation is needed on names; it is an infix operation with highest precedence, and is defined in Table 11.

Table 11 – Decorations of names in metalanguage

Notation	Definition
$i\ \mathit{decor}\ ^+$	the name that is like i but with the extra stroke $^+$

4.2.7 Tuples and Cartesian products

Tuples and Cartesian products are not primitive in Zermelo-Fraenkel set theory, but there are various ways in which they may be represented within that theory, such as the well-known encoding given by Kuratowski [1]. The choice of coding is irrelevant here and so is not specified. In this International Standard, particular names are always known either to have tuples or Cartesian products as their values, or not to have such values. Therefore there is never any possibility of accidental confusion between the encoding used to represent the tuple and any other value which is not a tuple.

In this mathematical metalanguage, tuples and Cartesian products with more than two components are interpreted as nested binary tuples and products, unlike in Z.

The syntactic forms are given in Table 12.

Table 12 – Tuples and Cartesian products in metalanguage

Notation	Name	Definition
(e_1, e_2)	pair	
$e_1 \mapsto e_2$	maplet	(e_1, e_2)
$\mathit{first}\ e$		$\mathit{first}(e_1, e_2) = e_1$
$\mathit{second}\ e$		$\mathit{second}(e_1, e_2) = e_2$
$e_1 \times e_2$		$\{i_1 : e_1;\ i_2 : e_2 \bullet (i_1, i_2)\}$
$(e_1, e_2, ..., e_n)$	tuple extension	$(e_1, (e_2, ..., e_n))$ where $n \geq 2$
$e_1 \times e_2 \times ... \times e_n$	Cartesian product	$e_1 \times (e_2 \times ... \times e_n)$ where $n \geq 2$
$e \uparrow n$	iterated product	$e \times ... \times e$ where there are $n \geq 2$ occurrences of e

The brackets delimiting a pair or tuple extension written with commas may not be omitted—they are not grouping parentheses.

4.2.8 Function comprehensions

Table 13 defines the notation for λ , which is a form of comprehension convenient when defining functions.

Table 13 – Function comprehensions in metalanguage

Notation	Definition
$\lambda\ i : e_1 \bullet e_2$	$\{i : e_1 \bullet i \mapsto e_2\}$
$\lambda\ i : e_1 \mid p \bullet e_2$	$\{i : e_1 \mid p \bullet i \mapsto e_2\}$
$\lambda\ i_1 : e_1 ; \ ...; \ i_n : e_n \bullet e$	$\{i_1 : e_1 ; \ ...; \ i_n : e_n \bullet (i_1, ..., i_n) \mapsto e\}$
$\lambda\ i_1 : e_1 ; \ ...; \ i_n : e_n \mid p \bullet e$	$\{i_1 : e_1 ; \ ...; \ i_n : e_n \mid p \bullet (i_1, ..., i_n) \mapsto e\}$

4.2.9 Relations

A relation is defined to be a set of Cartesian pairs. There are several operations involving relations, which are given equivalences in Table 14. A proposition about relations is given in Table 15.

Table 14 – Relations in metalanguage

Notation	Name	Definition
$id\ e$	identity function	$\lambda\ i : e \bullet i$
$dom\ e$	domain	$\{i : e \bullet first\ i\}$
$e^{\sim}$	relational inversion	$\{i : e \bullet second\ i \mapsto first\ i\}$
$e_1 \lhd e_2$	domain restriction	$\{i : e_2 \mid first\ i \in e_1\}$
$e_1 \ntriangleleft e_2$	domain subtraction	$\{i : e_2 \mid first\ i \notin e_1\}$
$e_1 (\!\| e_2 \|\!)$	relational image	$\{i : e_1 \mid first\ i \in e_2 \bullet second\ i\}$
$e_1 \fatsemi e_2$	relational composition	$\{i_1 : e_1 ; \ i_2 : e_2 \mid second\ i_1 = first\ i_2 \bullet first\ i_1 \mapsto second\ i_2\}$
$e_1 \oplus e_2$	relational overriding	$((dom\ e_2) \ntriangleleft e_1) \cup e_2$

Table 15 – Proposition about relations in metalanguage

Notation	Name	Definition
$e_1 \approx e_2$	compatible relations	$(dom\ e_2) \lhd e_1 = (dom\ e_1) \lhd e_2$

4.2.10 Functions

A function is a particular form of relation, where each domain element has only one corresponding range element. Table 16 shows the various forms of function that are identified, each being a set of functions.

Table 16 – Functions in metalanguage

Notation	Name	Definition
$e_1 \nrightarrow e_2$	functions	$\{i_1 : \mathbb{P}(e_1 \times e_2) \mid \forall\ i_2, i_3 : i_1 \mid first\ i_2 = first\ i_3 \bullet second\ i_2 = second\ i_3\}$
$e_1 \rightarrow e_2$	total functions	$\{i : e_1 \nrightarrow e_2 \mid dom\ i = e_1\}$
$e_1 \rightarrowtail\!\!\!\!\twoheadrightarrow e_2$	bijections	$\{i : e_1 \rightarrow e_2 \mid i^{\sim} \in e_2 \rightarrow e_1\}$
$e_1 \mathrel{+\!\!\!\twoheadrightarrow} e_2$	finite functions	$\{i : \mathbb{F}(e_1 \times e_2) \mid i \in e_1 \nrightarrow e_2\}$

4.2.10.1 Function use

A function can be juxtaposed with an argument to produce a result, using the notation of Table 17. Metalanguage notations introduced above that match the $e_1\ e_2$ pattern, such as $dom\ e$, are not applications in this sense.

Table 17 – Function use in metalanguage

Notation	Name	Definition
$e_1 \ e_2$	application	if there exists a unique e_3 such that $e_2 \mapsto e_3$ is in e_1, then the value of $e_1 \ e_2$ is e_3, otherwise each $e_1 \ e_2$ has a fixed but unknown value

4.2.11 Sequences

A sequence is a particular form of function, where the domain elements are all the natural numbers from 1 to the length of the sequence.

Table 18 – Sequences in metalanguage

Notation	Name	Definition
$\langle e_1, ..., e_n \rangle$	sequence	$\{1 \mapsto e_1, ..., n \mapsto e_n\}$

4.2.12 Disjointness

A labelled family of sets is disjoint when any distinct pair yields sets with no members in common.

Table 19 – Disjointness in metalanguage

Notation	Name	Definition
disjoint e	disjointness	$\forall e_1, e_2 : dom \ \ e \mid e_1 \neq e_2 \bullet e \ e_1 \cap e \ e_2 = \varnothing$

4.3 Transformation metalanguage

Each transformation rule is written in the following form.

$$\textit{concrete phrase template} \quad \Longrightarrow \quad \textit{less concrete phrase template}$$

The phrase templates are patterns; they are not specific sentences and they are not written in the syntactic metalanguage. These patterns are written in a notation based on the concrete and annotated syntaxes, with metavariables appearing in place of syntactically well-formed phrases. Where several phrases of the same syntactic classes have to be distinguished, these metavariables are given distinct numeric subscripts. The letters $_k$, $_m$, $_n$, $_r$ are used as metavariables for such numeric subscripts. The patterns can be viewed either as using the non-terminal symbols of the Z lexis with the -tok suffixes omitted from mathematical symbols, or as using the mathematical rendering with the box tokens in place of paragraph outlines. Transformations map parse trees of phrases to other parse trees. The metavariables are defined in Tables 20 and 21 (the phrases being defined in clauses 7 and 8, and the operator words in 7.4.4).

EXAMPLE 1 The syntactic transformation rule for a schema definition paragraph, and an informal reading of it, are as follows.

$$\texttt{SCH} \ i \ t \ \texttt{END} \quad \Longrightarrow \quad \texttt{AX} \ [i == t] \ \texttt{END}$$

A schema definition paragraph is formed from a box token `SCH`, a name i, a schema text t, and an `END` token. An equivalent axiomatic description paragraph is that which would be written textually as a box token `AX`, a [token, the original name i, a == token, the original schema text t, a] token, and an `END` token.

EXAMPLE 2 The semantic transformation rule for a schema hiding expression, and an informal reading of it, are as follows.

$$(e \ ⨾ \ \mathbb{P}[\sigma]) \setminus (i_1, ..., i_n) \quad \Longrightarrow \quad \exists \ i_1 : carrier \ (\sigma \ i_1); \ ...; \ i_n : carrier \ (\sigma \ i_n) \bullet e$$

Table 20 – Metavariables for phrases

Symbol	Definition
b	denotes a list of digits within a `NUMERAL` token.
c	denotes a digit within a `NUMERAL` token.
d	denotes a `Paragraph` phrase (d for definition/description).
de	denotes a `Declaration` phrase.
e	denotes an `Expression` phrase.
f	denotes a free type's `NAME` token.
g	denotes an injection's `NAME` token (g for in$g$ection).
h	denotes an element's `NAME` token (h for $h$element).
i, j	denote `NAME` tokens or `DeclName` or `RefName` phrases (i for identifier).
p	denotes a `Predicate` phrase.
s	denotes a `Section` phrase.
al	denotes an `ExpressionList` phrase (al for list argument).
t	denotes a `SchemaText` phrase (t for text).
u, v, w, x, y	denote distinct names for new local declarations.
z	denotes a `Specification` sentence.
τ	denotes a `Type` phrase.
σ	denotes a `Signature` phrase.
$^+$	denotes a `STROKE` token.
*	denotes a `{ STROKE }` phrase.
$\ldots$	denotes elision of repetitions of surrounding phrases, the total number of repetitions depending on syntax.

A schema with signature σ from which some names are hidden is semantically equivalent to the schema existential quantification of the hidden names from the schema. Each name is declared with the set that is the carrier set of the type of the name in the signature of the schema.

The applicability of a transformation rule can be guarded by a condition written above the $\Longrightarrow$ symbol. Local definitions can be associated with a transformation rule by appending a *where* clause, in which later definitions can refer to earlier definitions.

The transformation rule metalanguage is used in defining characterisation rules, syntactic transformation rules, type inference rules, and semantic transformation rules.

4.4 Type inference rule metalanguage

Each type inference rule is written in the following form.

$$\frac{\textit{type subsequents}}{\textit{type sequent}}(\textit{side–condition})$$

where local–declaration
and ...

This can be read as: if the type subsequents are valid, and the side-condition is true, then the type sequent is valid, in the context of the zero-or-more local declarations. The side-condition is optional; if omitted, the type inference rule is equivalent to one with a true side-condition.

The annotated syntax establishes notation for writing types as `Type` phrases and for writing signatures as `Signature` phrases. The $⨾$ operator allows annotations such as types to be associated with other phrases. Determining whether a type sequent is valid or not involves manipulation of types and signatures. This requires viewing types and signatures as values, and having a mathematical notation to do the manipulation. Signatures are viewed as functions from names to type values. `Type` is used to denote the set of type values as well as the set

Table 21 – Metavariables for operator words

Symbol	Definition
el	denotes an `EL` token.
elp	denotes an `ELP` token.
er	denotes an `ER` token.
ere	denotes an `ERE` token.
erep	denotes an `EREP` token.
erp	denotes an `ERP` token.
es	denotes an `ES` token.
ess	denotes an `ES` token or `SS` token.
in	denotes an `I` token.
ip	denotes an `IP` token or $\in$ token or $=$ token.
ln	denotes an `L` token.
lp	denotes an `LP` token.
post	denotes a `POST` token.
postp	denotes a `POSTP` token.
pre	denotes a `PRE` token.
prep	denotes a `PREP` token.
sr	denotes an `SR` token.
sre	denotes an `SRE` token.
srep	denotes an `SREP` token.
srp	denotes an `SRP` token.
ss	denotes an `SS` token.

of type phrases, the appropriate interpretation being distinguished by context of use. Similarly, `NAME` is also used to denote a set of name values. These values all lie within the type universe. A type's `NAME` has a corresponding type value in the type universe whereas its carrier set is in the semantic universe.

Type values are formed from just finite sets and ordered pairs, so the mathematical metalanguage introduced in section 4.2 suffices for their manipulation.

Details of which names are in scope are kept in environments. The various kinds of environment are defined in Table 22, and metavariables for environments are defined in Table 23.

Table 22 – Environments

Symbol	Definition
`TypeEnv`	denotes type environments, where $\mathtt{TypeEnv} == \mathtt{NAME} \mathrel{⇻} \mathtt{Type}$. Type environments associate names with types. They are like signatures, but are used in different contexts.
`SectTypeEnv`	denotes section-type environments, where $\mathtt{SectTypeEnv} == \mathtt{NAME} \mathrel{⇻} (\mathtt{NAME} \times \mathtt{Type})$. Section-type environments associate names of declarations with the name of the ancestral section that originally declared the name paired with its type.
`SectEnv`	denotes section environments, where $\mathtt{SectEnv} == \mathtt{NAME} \mathrel{⇻} \mathtt{SectTypeEnv}$. Section environments associate section names with section-type environments.

Type sequents are written using a $\vdash$ symbol superscripted with a mnemonic letter to distinguish the syntax of

Table 23 – Metavariables for environments

Symbol	Definition
Σ	denotes a type environment, $\Sigma : \texttt{TypeEnv}$.
Γ	denotes a section-type environment, $\Gamma : \texttt{SectTypeEnv}$.
Λ	denotes a section environment, $\Lambda : \texttt{SectEnv}$.

the phrase appearing to its right—see Table 24.

Table 24 – Type sequents

Symbol	Definition
$\vdash^{\mathcal{Z}} z$	a type sequent asserting that specification z is well-typed.
$\Lambda \vdash^{\mathcal{S}} s \,⨾\, \Gamma$	a type sequent asserting that, in the context of section environment Λ, section s has section-type environment Γ.
$\Sigma \vdash^{\mathcal{D}} d \,⨾\, \sigma$	a type sequent asserting that, in the context of type environment Σ, the paragraph d has signature σ.
$\Sigma \vdash^{\mathcal{P}} p$	a type sequent asserting that, in the context of type environment Σ, the predicate p is well-typed.
$\Sigma \vdash^{\mathcal{E}} e \,⨾\, \tau$	a type sequent asserting that, in the context of type environment Σ, the expression e has type τ.

NOTE 1 These superscripts are the same as the superscripts used on the $[\![\ ]\!]$ semantic brackets in the semantic relations below.

NOTE 2 Unlike the $\vdash ?$ symbol of Z, there is no `?` as these sequents are decidable.

The annotated phrases to the right of $\vdash$ in type sequents are phrase templates written using the same metavariables as the syntactic transformation rules; see Table 20.

EXAMPLE The type inference rule for a schema conjunction expression, and its informal reading, are as follows.

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \,⨾\, \tau_1 \qquad \Sigma \vdash^{\mathcal{E}} e_2 \,⨾\, \tau_2}{\Sigma \vdash^{\mathcal{E}} (e_1 \,⨾\, \tau_1) \wedge (e_2 \,⨾\, \tau_2) \,⨾\, \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta_1] \\ \tau_2 = \mathbb{P}[\beta_2] \\ \beta_1 \approx \beta_2 \\ \tau_3 = \mathbb{P}[\beta_1 \cup \beta_2] \end{array} \right)$$

In a schema conjunction expression $e_1 \wedge e_2$, expressions e_1 and e_2 shall be schemas, and their signatures shall be compatible. The type of the whole expression is that of the schema whose signature is the union of those of expressions e_1 and e_2.

NOTE 3 The metavariables σ_1 and σ_2 denote syntactic phrases. These are mapped implicitly to type values, so that the set union can be computed, and the resulting signature is implicitly mapped back to a syntactic phrase. These mappings are not made explicit as they would make the type inference rules harder to read, e.g. $[\![\, [\![\sigma_1]\!] \cup [\![\sigma_2]\!] \,]\!]$.

This metalanguage is used in defining type inference rules.

4.5 Semantic relation metalanguage

Most semantic relations are equations written in the following form.

$$[\![\mathit{phrase\ template}]\!] \quad = \quad \mathit{semantics}$$

Where the definition is only partial, the equality notation is not appropriate, and instead a lower bound is specified on the semantics.

$$\mathit{semantics} \quad \subseteq \quad [\![\ \mu\ e_1 \bullet e_2\ ]\!]^{\mathcal{E}}$$

The phrase templates use the same metavariables as used by the syntactic transformation rules—see Table 20.

Symbols concerned with the domain of the semantic definitions are listed in Tables 25 and 26.

Table 25 – Semantic universe

Symbol	Definition
$\mathbb{U}$	denotes the semantic universe, providing semantic values for all Z values, where $\mathbb{U} == \mathbb{W} \cup (\mathbb{W} \uparrow n \rightarrow \mathbb{W})$. $\mathbb{U}$ comprises $\mathbb{W}$ and Z generic definitions each as a function from the semantic values of its instantiating expressions (a tuple in $\mathbb{W}$) to a member of $\mathbb{W}$.
Model	denotes models, where $\mathit{Model} == \mathtt{NAME} \twoheadrightarrow \mathbb{U}$. Models associate names of declarations with semantic values. They are applied only to names in their domains, as guaranteed by well-typedness.
SectionModels	denotes functions from sections' names to their sets of models, where $\mathit{SectionModels} == \mathtt{NAME} \twoheadrightarrow \mathbb{P}\,\mathit{Model}$.

Table 26 – Variables over semantic universe

Symbol	Definition
M	denotes a model, $M : \mathit{Model}$.
T	denotes a section's name and its set of models, $T : \mathit{SectionModels}$.
t	denotes a binding semantic value, $t : \mathtt{NAME} \twoheadrightarrow \mathbb{W}$.
g	denotes a generic semantic value, $g : \mathbb{W} \rightarrow \mathbb{W}$.
w, x, y	denote non-generic semantic values, $w : \mathbb{W};\ x : \mathbb{W};\ y : \mathbb{W}$.

The meaning of a phrase template is given by a semantic relation from the Z phrase in terms of operations of ZF set theory on the semantic universe. There are different semantic relations for each syntactic notation, written using the conventional $[\![\ ]\!]$ semantic brackets, but here superscripted with a mnemonic letter to distinguish the syntax of phrase appearing within them—see Table 27.

NOTE The superscripts are the same as those used on the $\vdash$ of type sequents in the type inference rules above.

EXAMPLE The semantic relation for a conjunction predicate, and its informal reading, are as follows. The conjunction predicate $p_1 \wedge p_2$ is *true* if and only if p_1 and p_2 are *true*.

$$[\![\ p_1 \wedge p_2\ ]\!]^{\mathcal{P}} \quad = \quad [\![\ p_1\ ]\!]^{\mathcal{P}} \cap [\![\ p_2\ ]\!]^{\mathcal{P}}$$

In terms of the semantic universe, it is *true* in those models in which both p_1 and p_2 are *true*, and is *false* otherwise.

Within the semantic relations, the idioms listed in Table 28 occur repeatedly.

Semantic relation metalanguage is used in defining semantic relations.

Table 27 – Semantic relations

Symbol	Definition
$[\![z]\!]^{\mathcal{Z}}$	denotes the meaning of specification z, where $[\![z]\!]^{\mathcal{Z}} \in SectionModels$. The meaning of a specification is the function from its sections' names to their sets of models.
$[\![s]\!]^{\mathcal{S}}$	denotes the meaning of section s, where $[\![s]\!]^{\mathcal{S}} \in SectionModels \nrightarrow SectionModels$. The meaning of a section is given by the extension of a $SectionModels$ function with an extra maplet corresponding to the given section.
$[\![d]\!]^{\mathcal{D}}$	denotes the meaning of paragraph d, where $[\![d]\!]^{\mathcal{D}} \in Model \leftrightarrow Model$. The meaning of a paragraph relates a model to that model extended according to that paragraph.
$[\![p]\!]^{\mathcal{P}}$	denotes the meaning of predicate p, where $[\![p]\!]^{\mathcal{P}} \in \mathbb{P}\, Model$. The meaning of a predicate is the set of all models in which that predicate is true.
$[\![e]\!]^{\mathcal{E}}$	denotes the meaning of expression e, where $[\![e]\!]^{\mathcal{E}} \in Model \rightarrow \mathbb{W}$. The meaning of an expression is a function returning the semantic value of the expression in the given model.
$[\![\tau]\!]^{\mathcal{T}}$	denotes the meaning of type τ, where $[\![\tau]\!]^{\mathcal{T}} \in Model \nrightarrow \mathbb{P}\, \mathbb{U}$. The meaning of a type is the semantic value of its carrier set, as determined from the given model.

Table 28 – Semantic idioms

Idiom	Description
$[\![e]\!]^{\mathcal{E}} M$	denotes the value of expression e in model M
$M \oplus t$	denotes the model M giving semantic values for more global declarations overridden by the binding t giving the semantic values of locally declared names

5 Conformance

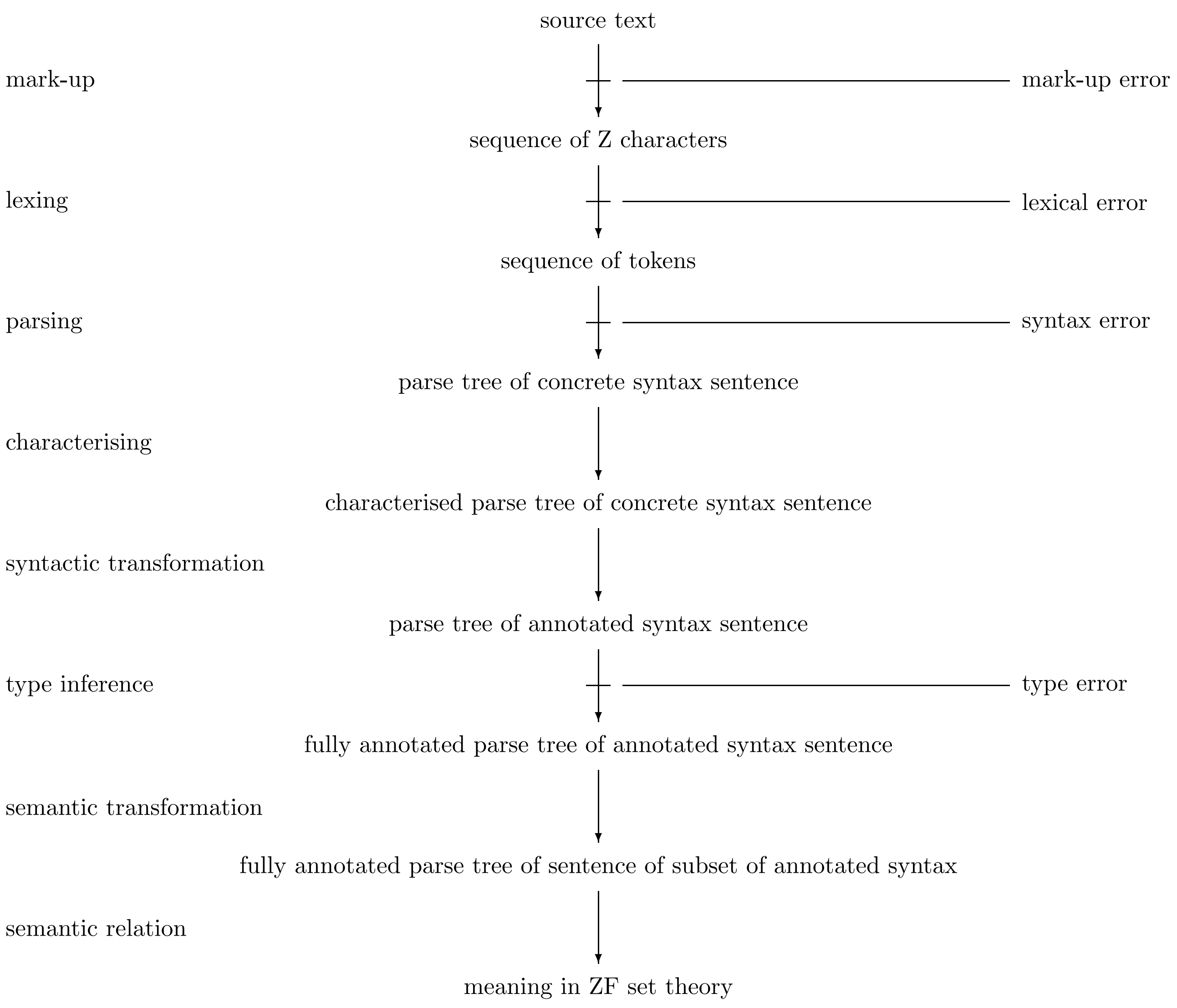
5.1 Phases of the definition

The definition of the Z notation is divided into a sequence of phases, as illustrated in Figure 1. Each arrow represents a phase from a representation of a Z specification at its source to another representation of the Z specification at its target. The phase is named at the left margin. Some phases detect errors in the specification; these are shown drawn off to the right-hand side.

NOTE 1 Figure 1 shows the order in which the phases are applied, and where errors are detected; it does not show information flows.

NOTE 2 The arrows are analogous to total and partial function arrows in the Z mathematical toolkit, but drawn vertically.

Figure 1 – Phases of the definition



5.2 Conformance requirements

5.2.1 Specification conformance

For a Z specification to conform to this International Standard, no errors shall be detected by any of the phases shown in Figure 1. In words, for a Z specification to conform to this International Standard, its formal text shall be valid mark-up of a sequence of Z characters, that can be lexed as a valid sequence of tokens, that can be parsed as a sentence of the concrete syntax, and that is well-typed according to the type inference system.

NOTE The presence of sections that have no models does not affect the conformance of their specification.

5.2.2 Mark-up conformance

A mark-up for Z based on LaTeX [9] conforms to this International Standard if and only if it follows the rules given for LaTeX mark-up in A.2.

A mark-up for Z used in email correspondence conforms to this International Standard if and only if it follows the rules given for email mark-up in A.3.

Mark-up for Z based on any other mark-up language is permitted; it shall be possible to define a functional mapping from that mark-up to sequences of Z characters.

The ISO/IEC 10646 representation may be used directly—the identity function is an acceptable mapping.

5.2.3 Deductive system conformance

A Z deductive system conforms to this International Standard if and only if its rules are sound with respect to the semantics, i.e. if both of the following conditions hold:

a) all of its axioms hold in all models of all Z specifications, i.e. for any axiom p,

$$[\![\ p \ ]\!]^{\mathcal{P}} = Model$$

b) all of its rules of inference have the property that the intersection of the sets of models of each of the premises is contained in the model of the conclusion, i.e. for any rule of inference where p is deduced from p_1, ... p_n,

$$[\![\ p_1 \ ]\!]^{\mathcal{P}} \cap ... \cap [\![\ p_n \ ]\!]^{\mathcal{P}} \subseteq [\![\ p \ ]\!]^{\mathcal{P}}$$

All constraints appearing before a conjecture in a specification may be used as premises in inferences about that conjecture. A Z deductive system should document whether or not it allows constraints appearing after a conjecture in a specification to be used as premises in inferences about that conjecture.

The semantic relation is defined loosely in this International Standard, so as to permit alternative treatments of undefinedness. A Z deductive system may take a particular position on undefinedness. That position should be clearly documented.

5.2.4 Mathematical toolkit conformance

A Z section whose name is the same as a section of the mathematical toolkit of annex B conforms to this International Standard if and only if it defines the same set of models as that section of the mathematical toolkit. A Z section whose name is *prelude* conforms to this International Standard if and only if it defines the same set of models as the section in clause 11.

A mathematical toolkit conforms to this standard if it defines a conformant section called standard_toolkit.

NOTE 1 The set of models defined by a section within a specification may be found by applying the meaning of the specification to the section's name.

NOTE 2 A conforming section of the toolkit may formulate its definitions differently from those in annex B.

NOTE 3 A conforming section of the toolkit may partition its definitions amongst parent sections that differ from those in annex B.

NOTE 4 Alternative and additional toolkits are not precluded, but are required to have different section names to avoid confusion.

NOTE 5 Some names are loosely defined, such as $\mathbb{A}$, and may be further constrained by sections that use toolkit sections, but not by toolkit sections themselves.

5.2.5 Support tool conformance

A strongly conforming Z support tool shall recognise at least one conforming mark-up, accepting all conforming Z specifications presented to it, and rejecting all non-conforming Z specifications presented to it. A weakly conforming Z support tool shall never accept a non-conforming Z specification, nor reject a conforming Z specification, but it may state that it is unable to determine whether or not a Z specification conforms.

NOTE Strong conformance can be summarised as always being right, whereas weak conformance is never being wrong.

EXAMPLE A tool would be weakly conformant if it were to announce its inability to determine the conformance of a Z specification that used names longer than the tool could handle, but would be non-conformant if it silently truncated long names.

Certain exceptions to general rules are anticipated and permitted in subsequent clauses, because of, for example, implementation considerations or for backwards compatibility with pre-existing tools.

5.3 Structure of this document

The phases in the definition of the Z notation, and the representations of specifications manipulated by those phases, as illustrated in Figure 1, are specified in the following clauses and annexes.

Annex A, Mark-ups, specifies two source text representations and corresponding mark-up phases for translating source text to sequences of Z characters.

Clause 6 specifies the Z characters by their appearances and their names in ISO/IEC 10646.

Clause 7, Lexis, specifies tokens and the lexing phase that translates a sequence of Z characters to a sequence of tokens.

Clause 8 specifies the grammar of the concrete syntax, and hence abstractly specifies the parsing phase that translates a sequence of tokens to a parse tree of a concrete syntax sentence. Some information from parsing operator template paragraphs is fed back to the lexis phase.

Clause 9 specifies the characterising phase, during which characteristic tuples are made explicit in the parse tree of a concrete syntax sentence.

Clause 10 specifies the grammar of the annotated syntax, defining the target language for the syntactic transformation phase.

Clause 11 specifies the prelude section, providing an initial environment of definitions.

Clause 12 specifies the syntactic transformation phase that translates a parse tree of a concrete syntax sentence to a parse tree of an equivalent annotated syntax sentence.

Clause 13 specifies the type inference phase, during which type annotations are added to the parse tree of the annotated syntax sentence, and reference expressions that refer to generic definitions are translated to generic instantiation expressions.

Clause 14 specifies the semantic transformation phase, during which some annotated parse trees are translated to equivalent other annotated parse trees.

Clause 15 specifies the semantic relation between a sentence of the remaining annotated syntax and its meaning in ZF set theory.

Annex C duplicates those parts of the definition that fit into an organisation by concrete syntax production.

6 Z characters

6.1 Introduction

A Z character is the smallest unit of information in this International Standard; Z characters are used to build tokens (clause 7), which are in turn the units of information in the concrete syntax (clause 8). The Z characters are defined by reference to ISO/IEC 10646-1 and ISO/IEC 10646-2: for each Z character is listed its appearance, code position and name.

Many Z characters are not present in the standard 7-bit ASCII encoding [4]. It is possible to represent Z characters in ASCII, by defining a mark-up, where several ASCII characters are used together to represent a single Z character. This International Standard defines some ASCII mark-ups in annex A by relation to the ISO/IEC 10646 representation defined here. Other mark-ups of Z characters can similarly be defined by relation to the ISO/IEC 10646 representation.

6.2 Formal definition of Z characters

```
ZCHAR          = DIGIT | LETTER | SPECIAL | SYMBOL ;

DIGIT          = '0' | '1' | '2' | '3' | '4' | '5' | '6' | '7' | '8' | '9'
               | any other ISO/IEC 10646 character with the 'decimal' property (as supported)
               ;

LETTER         = LATIN | GREEK | OTHERLETTER
               | any characters of the mathematical toolkit with 'letter' property (as supported)
               | any other ISO/IEC 10646 characters with the 'letter' property (as supported)
               ;

LATIN          = 'A' | 'B' | 'C' | 'D' | 'E' | 'F' | 'G' | 'H' | 'I'
               | 'J' | 'K' | 'L' | 'M' | 'N' | 'O' | 'P' | 'Q' | 'R'
               | 'S' | 'T' | 'U' | 'V' | 'W' | 'X' | 'Y' | 'Z'
               | 'a' | 'b' | 'c' | 'd' | 'e' | 'f' | 'g' | 'h' | 'i'
               | 'j' | 'k' | 'l' | 'm' | 'n' | 'o' | 'p' | 'q' | 'r'
               | 's' | 't' | 'u' | 'v' | 'w' | 'x' | 'y' | 'z'
               ;

GREEK          = 'Δ' | 'Ξ' | 'θ' | 'λ' | 'μ' ;

OTHERLETTER    = '𝔸' | 'ℕ' | 'ℙ' ;

SPECIAL        = STROKECHAR | WORDGLUE | BRACKET | BOXCHAR | SPACE ;

STROKECHAR     = '′' | '!' | '?' ;

WORDGLUE       = '↗' | '↙' | '↘' | '↖' | '_' ;

BRACKET        = '(' | ')' | '[' | ']' | '{' | '}' | '⦉' | '⦊' | '⟪' | '⟫' ;

BOXCHAR        = AXCHAR | SCHCHAR | GENCHAR | ENDCHAR | NLCHAR ;

SYMBOL         = '|' | '&' | '⊢' | '∧' | '∨' | '⇒' | '⇔' | '¬' | '∀' | '∃'
               | '×' | '/' | '=' | '∈' | ':' | ';' | ',' | '.' | '•'
               | '\' | '⁀' | '⨾' | '≫' | '+'
               | any characters of the mathematical toolkit not included above (as supported)
               | any other ISO/IEC 10646 characters not included above (as supported)
               ;
```


6.3 Additional restrictions and notes

The word supported means available for use in presenting a specification.

The characters enumerated in the formal definition are those used by the core language; they shall be supported. If the mathematical toolkit is supported, then its characters shall be supported. The "other ISO/IEC 10646 characters" may also be supported, extending `DIGIT`, `LETTER` or `SYMBOL` according to their property, but not extending `SPECIAL`. Use of characters that are absent from ISO/IEC 10646 is permitted, but there is no standard way of distinguishing which of `DIGIT`, `LETTER` or `SYMBOL` (not `SPECIAL`) they extend, and specifications using them might not be interchangeable between tools.

`SPACE` is a Z character that serves to separate two sequences of Z characters that would otherwise be mis-lexed as a single token.

NOTE 1 `STROKECHAR` characters are used in `STROKE` tokens in the lexis.

NOTE 2 `WORDGLUE` characters are used in building `NAME` tokens in the lexis.

NOTE 3 `BOXCHAR` characters correspond to Z's distinctive boxes around paragraphs.

6.4 Z character representations

6.4.1 Introduction

The following tables show the Z characters in their mathematical representation. (Other representations are given in annex A.) The columns give:

Math: The representation for rendering the character on a high resolution device, such as a bit-mapped screen, or on paper (either hand-written, or printed).

Code position: The encoding of the Z character in ISO/IEC 10646. Encodings that are enclosed in brackets are ones that have been assigned by the Unicode Technical Committee to appear in a future version of their standard [5], but which are not yet known to have been adopted into ISO/IEC 10646.

Character name: The name for the character in ISO/IEC 10646 or, in the case of characters with bracketed code positions, the name suggested by the Unicode Technical Committee.

NOTE 1 The code position column is included to assist location of the characters by many users; it is not necessary for definition of the characters.

NOTE 2 From the point of view of conformance, only those Z characters with code positions already registered in ISO/IEC 10646 are relevant. When the other Z characters are adopted into ISO/IEC 10646, a Technical Corrigendum to the present standard will be issued.

6.4.2 Digit characters

Math	Code position	Character name
0	0000 0030	DIGIT ZERO
⋮	⋮	⋮
9	0000 0039	DIGIT NINE

6.4.3 Letter characters

6.4.3.1 Latin alphabet characters

Math	Code position	Character name
A	0000 0041	LATIN CAPITAL LETTER A
⋮	⋮	⋮
Z	0000 005A	LATIN CAPITAL LETTER Z
a	0000 0061	LATIN SMALL LETTER A
⋮	⋮	⋮
z	0000 007A	LATIN SMALL LETTER Z

6.4.3.2 Greek alphabet characters

The Greek alphabet characters used by the core language are those listed here.

Math	Code position	Character name
Δ	0000 0394	GREEK CAPITAL LETTER DELTA
Ξ	0000 039E	GREEK CAPITAL LETTER XI
θ	0000 03B8	GREEK SMALL LETTER THETA
λ	0000 03BB	GREEK SMALL LETTER LAMBDA
μ	0000 03BC	GREEK SMALL LETTER MU

6.4.3.3 Other Z core language letter characters

The other Z core language characters with the ISO/IEC 10646 'letter' property are listed here. (These characters are introduced in the prelude, clause 11.)

Math	Code position	Character name
$\mathbb{A}$	[0001 D538]	MATH OPEN-FACE CAPITAL A
$\mathbb{N}$	0000 2115	DOUBLE-STRUCK CAPITAL N
$\mathbb{P}$	0000 2119	DOUBLE-STRUCK CAPITAL P

6.4.4 Special characters

6.4.4.1 Stroke characters

Math	Code position	Character name
$'$	0000 02B9	MODIFIER LETTER PRIME
!	0000 0021	EXCLAMATION MARK
?	0000 003F	QUESTION MARK

6.4.4.2 Word glue characters

The characters '$\nearrow$', '$\swarrow$', '$\searrow$', and '$\nwarrow$' may be presented as in-line literals, or they may indicate a raising/lowering of the text, and possible size change. Such rendering details are not defined here.

Math	Code position	Character name
$\nearrow$	0000 2197	NORTH EAST ARROW
$\swarrow$	0000 2199	SOUTH WEST ARROW
$\searrow$	0000 2198	SOUTH EAST ARROW
$\nwarrow$	0000 2196	NORTH WEST ARROW
_	0000 005F	LOW LINE

6.4.4.3 Bracket characters

Math	Code position	Character name
(	0000 0028	LEFT PARENTHESIS
)	0000 0029	RIGHT PARENTHESIS
[	0000 005B	LEFT SQUARE BRACKET
]	0000 005D	RIGHT SQUARE BRACKET
{	0000 007B	LEFT CURLY BRACKET
}	0000 007D	RIGHT CURLY BRACKET
⦉	[0000 2989]	Z NOTATION LEFT BINDING BRACKET
⦊	[0000 298A]	Z NOTATION RIGHT BINDING BRACKET
《	0000 300A	LEFT DOUBLE ANGLE BRACKET
》	0000 300B	RIGHT DOUBLE ANGLE BRACKET

6.4.4.4 Box characters

The `ENDCHAR` character is used to mark the end of a `Paragraph`. The `NLCHAR` character is used to mark a hard newline (see section 7.5). The box rendering of the `BOXCHAR` characters is as lines drawn around the Z text (see section 8.5).

Z character	Simple rendering	Code position	Character name
`AXCHAR`	╷	0000 2577	BOX DRAWINGS LIGHT DOWN
`SCHCHAR`	┌	0000 250C	BOX DRAWINGS LIGHT DOWN AND RIGHT
`GENCHAR`	═	0000 2550	BOX DRAWINGS DOUBLE HORIZONTAL
`ENDCHAR`	(new line)	0000 2029	PARAGRAPH SEPARATOR
`NLCHAR`	(new line)	0000 2028	LINE SEPARATOR

6.4.4.5 `SPACE` character

Z character	Code position	Character name
`SPACE`	0000 0020	SPACE

6.4.5 Symbol characters except mathematical toolkit characters

Math	Code position	Character name
$\vert$	0000 007C	VERTICAL LINE
&	0000 0026	AMPERSAND
$\vdash$	0000 22A2	RIGHT TACK
$\wedge$	0000 2227	LOGICAL AND
$\vee$	0000 2228	LOGICAL OR
$\Rightarrow$	0000 21D2	RIGHTWARDS DOUBLE ARROW
$\Leftrightarrow$	0000 21D4	LEFT RIGHT DOUBLE ARROW
$\neg$	0000 00AC	NOT SIGN
$\forall$	0000 2200	FOR ALL
$\exists$	0000 2203	THERE EXISTS
$\times$	0000 00D7	MULTIPLICATION SIGN
$/$	0000 002F	SOLIDUS
$=$	0000 003D	EQUALS SIGN
$\in$	0000 2208	ELEMENT OF
:	0000 003A	COLON
;	0000 003B	SEMICOLON
,	0000 002C	COMMA
.	0000 002E	FULL STOP
$\bullet$	[0000 2981]	Z NOTATION SPOT
$\big\backslash$	[0000 2055]	BIG REVERSE SOLIDUS
$\upharpoonright$	[0000 2A21]	Z NOTATION SCHEMA PROJECTION
$\fatsemi$	[0000 2A1F]	Z NOTATION SCHEMA COMPOSITION
$\gg$	[0000 2A20]	Z NOTATION SCHEMA PIPING
$+$	0000 002B	PLUS SIGN
⦂	[0000 2982]	Z NOTATION TYPE COLON

6.4.6 Mathematical toolkit characters

The mathematical toolkit (annex B) need not be supported by an implementation. If it is supported, it shall use the representations given here.

Mathematical toolkit names that use only Z core language characters, or combinations of Z characters defined here, are not themselves listed here.

Math	Code position	Character name
↔	0000 2194	LEFT RIGHT ARROW
→	0000 2192	RIGHTWARDS ARROW
≠	0000 2260	NOT EQUAL TO
∉	0000 2209	NOT AN ELEMENT OF
∅	0000 2205	EMPTY SET
⊆	0000 2286	SUBSET OF OR EQUAL TO
⊂	0000 2282	SUBSET OF
∪	0000 222A	UNION
∩	0000 2229	INTERSECTION
\	0000 005C	REVERSE SOLIDUS
⊖	0000 2296	CIRCLED MINUS
⋃	0000 22C3	N-ARY UNION
⋂	0000 22C2	N-ARY INTERSECTION
𝔽	[0001 D53D]	MATH OPEN-FACE CAPITAL F
↦	0000 21A6	RIGHTWARDS ARROW FROM BAR
⨾	[0000 2A3E]	Z NOTATION RELATIONAL COMPOSITION
∘	0000 2218	RING OPERATOR
◁	0000 25C1	WHITE LEFT-POINTING TRIANGLE
▷	0000 25B7	WHITE RIGHT-POINTING TRIANGLE
⩤	[0000 2A64]	Z NOTATION DOMAIN ANTIRESTRICTION
⩥	[0000 2A65]	Z NOTATION RANGE ANTIRESTRICTION
~	0000 007E	TILDE
⦇	[0000 2987]	Z NOTATION LEFT IMAGE BRACKET
⦈	[0000 2988]	Z NOTATION RIGHT IMAGE BRACKET
⊕	0000 2295	CIRCLED PLUS
⇸	[0000 21F8]	RIGHTWARDS ARROW WITH VERTICAL STROKE
⤔	[0000 2914]	RIGHTWARDS ARROW WITH TAIL WITH VERTICAL STROKE
↣	0000 21A3	RIGHTWARDS ARROW WITH TAIL
⤀	[0000 2900]	RIGHTWARDS TWO-HEADED ARROW WITH VERTICAL STROKE
↠	0000 21A0	RIGHTWARDS TWO-HEADED ARROW
⤖	[0000 2916]	RIGHTWARDS TWO-HEADED ARROW WITH TAIL
⇻	[0000 21FB]	RIGHTWARDS ARROW WITH DOUBLE VERTICAL STROKE
⤕	[0000 2915]	RIGHTWARDS ARROW WITH TAIL WITH DOUBLE VERTICAL STROKE
ℤ	0000 2124	DOUBLE-STRUCK CAPITAL Z
-	0000 002D	HYPHEN-MINUS
−	0000 2212	MINUS SIGN
≤	0000 2264	LESS-THAN OR EQUAL TO
<	0000 003C	LESS-THAN SIGN
≥	0000 2265	GREATER-THAN OR EQUAL TO
>	0000 003E	GREATER-THAN SIGN
#	0000 0023	NUMBER SIGN
〈	0000 2329	LEFT-POINTING ANGLE BRACKET
〉	0000 232A	RIGHT-POINTING ANGLE BRACKET
⁀	0000 2040	CHARACTER TIE
↿	0000 21BF	UPWARDS HARPOON WITH BARB LEFTWARDS
↾	0000 21BE	UPWARDS HARPOON WITH BARB RIGHTWARDS

6.4.7 Renderings of Z characters

Renderings of Z characters are called glyphs (following the terminology of ISO/IEC 10646). A rendering of a Z character on a graphics screen is typically different from its rendering on a piece of paper: the glyphs used for Z characters are device-dependent.

A Z character may also be rendered using different glyphs at different places in a specification, for reasons of emphasis or aesthetics, but such different glyphs still represent the same Z character. For example, '*d*', 'd', 'd' and '**d**' are all the same Z character.

For historical reasons, some different Z characters have similar-looking renderings. In particular:

- schema composition '⨾' and the mathematical toolkit character relational composition '⨾' are different Z characters;
- schema projection '↾' and the mathematical toolkit character filter '↾' are different Z characters;
- schema hiding '\' and the mathematical toolkit character set minus '\' are different Z characters.

7 Lexis

7.1 Introduction

The lexis specifies a function from sequences of Z characters to sequences of tokens. The domain of the function involves all the Z characters of clause 6. The range of the function involves all the tokens used in clause 8. The function is partial: sequences of Z characters that do not conform to the lexis are excluded from consideration at this stage.

The lexis is composed of two parts: a context-free part followed by a context-sensitive part. The former translates the stream of Z characters into a stream of DECORWORDs and tokens. The latter classifies each DECORWORD as being a keyword, an operator token, or a NAME token, taking into account the lexical scopes of the operators (see 7.4.4).

7.2 Formal definition of context-free lexis

```
TOKENSTREAM     = { SPACE } , { TOKEN , { SPACE } } ;

TOKEN           = DECORWORD | NUMERAL | STROKE
                | (-tok | )-tok | [-tok | ]-tok | {-tok | }-tok | ⦉ | ⦊ | ⟪ | ⟫
                | AX | GENAX | SCH | GENSCH | END | NL
                ;

DECORWORD       = WORD , { STROKE } ;

WORD            = WORDPART , { WORDPART }
                | LETTER , ALPHASTR , { WORDPART }
                | SYMBOL , SYMBOLSTR , { WORDPART }
                ;

WORDPART        = WORDGLUE , ( ALPHASTR | SYMBOLSTR ) ;
ALPHASTR        = { LETTER | DIGIT } ;
SYMBOLSTR       = { SYMBOL } ;

NUMERAL         = NUMERAL , DIGIT
                | DIGIT
                ;

STROKE          = STROKECHAR
                | '↘' , DIGIT , '↖'
                ;
```



```
(-tok     = '(' ;
)-tok     = ')' ;
[-tok     = '[' ;
]-tok     = ']' ;
{-tok     = '{' ;
}-tok     = '}' ;
⦉         = '⦉' ;
⦊         = '⦊' ;
⟪         = '⟪' ;
⟫         = '⟫' ;

AX        = AXCHAR ;
SCH       = SCHCHAR ;
GENAX     = AXCHAR , GENCHAR ;
GENSCH    = SCHCHAR , GENCHAR ;
END       = ENDCHAR ;
NL        = NLCHAR ;
```

7.3 Additional lexical restrictions, notes and examples

SPACE is not itself lexed as part of any token. It can be used freely between tokens. The cases where its use is necessary are: between two WORDs, which would otherwise be lexed as a single WORD; between an alphabetic WORD and a NUMERAL, which would otherwise be lexed as a single WORD; between a DECORWORD and a STROKE (a decoration expression), which would otherwise be lexed as a single DECORWORD; and between two consecutive NUMERALs, which would otherwise be lexed as a single NUMERAL.

Words are formed from alphanumeric and symbolic parts.

EXAMPLE 1 The following strings of Z characters are single DECORWORDs: '&+=', 'x_+_y', 'x_+y', 'x^+y', '$x^+{}_+y$'. However, '$x+y$' comprises the three DECORWORDs 'x', '+' and 'y'.

EXAMPLE 2 The following strings of Z characters are single DECORWORDs: 'λS', 'ΔS', '$\exists\times$', '$\exists$_X', '$\exists_X$'. However, '$\exists X$' is the keyword token '$\exists$', followed by the DECORWORD 'X'.

EXAMPLE 3 The following strings of Z characters are single DECORWORDs: '$\times$:$\in$', 'x_:_e', '${}_{x:e}$'. However, 'x:e' is the word 'x', followed by the keyword token ':', followed by the DECORWORD 'e'.

The concrete syntax allows a RefName, which may be a NAME that includes strokes; it also allows an expression to be decorated with a stroke. When the decorated expression is a NAME, the lexis disambiguates the two cases by using the white space between the DECORWORD and the STROKE: in the absence of any white space, the stroke shall be lexed as part of the DECORWORD; in the presence of white space, the stroke shall be lexed as a decoration on the expression.

EXAMPLE 4 x! is the DECORWORD comprising the WORD 'x' followed by the STROKE '!'.
x ! is the decorated expression comprising the RefName expression 'x' decorated with the STROKE '!'.
x! ! is the decorated expression comprising the RefName expression 'x!' decorated with the STROKE '!'.

The lexis allows a WORD to include subscript digits; it also allows a DECORWORD to be decorated with subscript digits. Trailing subscript digits shall be lexed as strokes, not as part of a WORD.

EXAMPLE 5 $x_{a\,1}$ is a DECORWORD comprising the WORD 'x_a' and the STROKE '${}_1$'.
x_a? is a DECORWORD comprising the WORD 'x_a' and the STROKE '?'.
$x_{1\,a}$ is a DECORWORD comprising the WORD '$x_{1\,a}$' and no strokes.

A multi-digit last WORDPART enclosed in a $\searrow \ldots \nwarrow$ pair is deprecated, because of the visual ambiguity with multiple STROKE subscript digits.

NOTE 1 There is no need for any particular mark-up to follow these rules; this syntax applies only to tokens built from Z characters.

NOTE 2 Although a parser does not need to know the spelling of particular instances of `DECORWORD`, `NUMERAL`, `STROKE`, etc tokens, subsequent phases of processing such as typechecking do. This relation between instances of tokens and spellings is not explicitly formalized here.

Some tools may impose restrictions on the forms of some names. Section names that are entirely alphanumeric, capitalized and short are the most likely to be portable between tools.

7.4 Context-sensitive lexis

7.4.1 Introduction

The context-sensitive part of lexis maps each `DECORWORD` to either a keyword token, an operator token, or a `NAME` token. It also strips all `SPACE`s from the token stream, and forwards all other tokens unchanged.

If a `DECORWORD`'s spelling is exactly that of a keyword, the `DECORWORD` is mapped to the corresponding keyword token. Otherwise, if the `DECORWORD`'s `WORD` part's spelling is that of an operator word, the `DECORWORD` is mapped to the relevant operator token. Otherwise, the `DECORWORD` is mapped to a `NAME` token.

In the case of a `NAME` token, for every '↘' `WORDGLUE` character in its `WORD` part, there shall be a paired following '↖' `WORDGLUE` character, for every '↗' `WORDGLUE` character in its `WORD` part, there shall be a paired following '↙' `WORDGLUE` character, and these shall occur only in nested pairs.

NOTE 1 Operators have a similar restriction applied to the whole operator name (12.2.8), not to the individual words within the operator.

The keywords are as listed in the following tables. No other spellings give rise to keyword tokens. The columns give:

Spelling: The sequence of Z characters representing the rendering of the token on a high resolution device, such as a bit-mapped screen, or on paper (either hand-written, or printed).

Token: The token used for that keyword in the concrete syntax.

Token name: A suggested form for reading the keyword out loud, suitable for use in reviews, or for discussing specifications over the telephone. In the following, an English language form is given; for other natural languages, other forms may be defined.

NOTE 2 Even where a keyword consists of a single Z character, the token name tends to reflect the keyword's function rather than the form of the Z character.

7.4.2 Alphabetic keywords

Spelling	**Token**	**Token name**
else	else	else
false	false	false
function	function	function
generic	generic	generic
if	if	if
leftassoc	leftassoc	left [associative]
let	let	let
$\mathbb{P}$	$\mathbb{P}$	powerset
parents	parents	parents
pre	pre	pre[condition]
relation	relation	relation
rightassoc	rightassoc	right [associative]
section	section	section
then	then	then
true	true	true

7.4.3 Symbolic keywords

Spelling	**Token**	**Token name**
:	:	colon
==	==	define equal
,	,-tok	comma
::=	::=	free equals
|	|-tok	bar
&	&	and also [free types]
$\setminus$	$\setminus$	hide
/	/	rename
.	.	select | dot
;	; -tok	semi[colon]
_	_	arg[ument]
,,	,,	list arg[ument]
=	=-tok	equals

EXAMPLE 1 = is recognised as the keyword token =-tok; := is recognised as a `NAME` token; ::= is recognised as the keyword token.

Spelling	**Token**	**Token name**
$\vdash ?$	$\vdash ?$	conjecture
$\forall$	$\forall$	for all
$\bullet$	$\bullet$	spot | fat dot
$\exists$	$\exists$	exists
$\exists_1$	$\exists_1$	unique exists
$\Leftrightarrow$	$\Leftrightarrow$	equivalent | if and only if
$\Rightarrow$	$\Rightarrow$	implies
$\lor$	$\lor$	or
$\land$	$\land$	and
$\lnot$	$\lnot$	not
$\in$	$\in$	in | member of | element of
$\upharpoonright$	$\upharpoonright$	project
$\times$	$\times$	cross
λ	λ	lambda
μ	μ	mu
θ	θ	theta
$\fatsemi$	$\fatsemi$	schema compose
$\gg$	$\gg$	schema pipe

EXAMPLE 2 $\exists$ and $\exists_1$ ('$\exists$', '$\searrow$', '1', '$\nwarrow$') are recognised as keyword tokens; $\exists_0$ ('$\exists$', '$\searrow$', '0', '$\nwarrow$') is recognised as a `NAME` token.

EXAMPLE 3 λ is recognised as the keyword token; λx is recognised as a `NAME` token; $\lambda\ x$ is recognised as the keyword token followed by a `NAME` token.

7.4.4 User-defined operators

Each operator template creates additional keyword-like associations between `WORD`s and operator tokens. The scope of these associations is the whole of the section in which the operator template appears (not just from the operator template onwards), as well as all sections of which that section is an ancestor, excluding section headers.

NOTE The set of active associations is always a function. This International Standard does not specify how that function is determined: operator template paragraphs provide the information, yet in their concrete syntax it is assumed that the function is already known.

The appropriate token for an operator word is as follows.

`PREP`	prefix unary relation
`PRE`	prefix unary function or generic
`POSTP`	postfix unary relation
`POST`	postfix unary function or generic
`IP`	infix binary relation
`I`	infix binary function or generic
`LP`	left bracket of non-unary relation
`L`	left bracket of non-unary function or generic
`ELP`	first word preceded by expression of non-unary relation
`EL`	first word preceded by expression of non-unary function or generic
`ERP`	right bracket preceded by expression of non-unary relation
`ER`	right bracket preceded by expression of non-unary function or generic
`SRP`	right bracket preceded by list argument of non-unary relation
`SR`	right bracket preceded by list argument of non-unary function or generic
`EREP`	last word followed by expression and preceded by expression of tertiary or higher relation
`ERE`	last word followed by expression and preceded by expression of tertiary or higher function or generic
`SREP`	last word followed by expression and preceded by list argument of tertiary or higher relation
`SRE`	last word followed by expression and preceded by list argument of tertiary or higher function or generic
`ES`	middle word preceded by expression of non-unary operator
`SS`	middle word preceded by list argument of non-unary operator

EXAMPLE 1 The operator template paragraph for the (_ + _) operator adds one entry to the mapping.

Spelling	**Token**
+	`I`

EXAMPLE 2 The operator template paragraph for the (_ ⦇ _ ⦈) operator adds two entries to the mapping.

Spelling	**Token**
⦇	`EL`
⦈	`ER`

EXAMPLE 3 The operator template paragraph for the (*disjoint* _) operator adds one entry to the mapping.

Spelling	**Token**
disjoint	`PREP`

EXAMPLE 4 The operator template paragraph for the (⟨ _ ⟩) operator adds two entries to the mapping.

Spelling	**Token**
⟨	`L`
⟩	`SR`

7.5 Newlines

The Z character `NLCHAR` is lexed either as a token separator (like the `SPACE` character) or as the token `NL`, depending on its context. A *soft newline* is a `NLCHAR` that is lexed as a token separator. A *hard newline* is a `NLCHAR` that is lexed as a `NL` token.

Tokens are assigned to a *newline category*, namely BOTH, AFTER, BEFORE or NEITHER, based on whether that token could start or end a Z phrase.

- BOTH: newlines are soft before and after the token, because it is infix, something else has to appear before it and after it.

  else function generic leftassoc parents relation rightassoc section then
  $::=$ $|$-tok $\langle\!\langle$ $\rangle\!\rangle$ & $\vdash?$, , $\land$ $\lor$ $\Rightarrow$ $\Leftrightarrow$ $\times$ / =-tok $\in$ == : ; -tok ,-tok . $\bullet$ $\setminus$ $\upharpoonright$ $\fatsemi$ $\gg$
  `I IP EL ELP ERE EREP ES SS SRE SREP`

  All newlines are soft outside of a `DeclPart` or a `Predicate`.

  NOTE Tokens that cannot appear in these contexts are in category BOTH. This includes the box tokens. Newlines at the very beginning or end of a specification are soft.

- AFTER: newlines are soft after the token, because it is prefix, something else has to appear after it.

  if let pre
  [−tok _ $\neg$ $\forall$ $\exists$ $\exists_1$ $\mathbb{P}$ (−tok {−tok $\langle\!|$ λ μ θ
  `PRE PREP L LP`

- BEFORE: newlines are soft before the token, because it is postfix, something else has to appear before it.

  ]−tok)−tok }−tok $|\!\rangle$
  `POST POSTP ER ERP SR SRP`

- NEITHER: no newlines are soft, because such a token is nofix, nothing else need appear before or after it.

  false true
  `NAME NUMERAL STROKE`

For each `NLCHAR`, the newline categories of the closest token generated from the preceding Z characters and the token generated from the immediately following Z characters are examined. If either token allows the newline to be soft in that position, it is soft, otherwise it is hard (and hence recognised as a `NL` token).

The operator template paragraph allows the definition of various mixfix names (see section 7.4.4), which are placed in the appropriate newline category. Other (ordinary) user declared names are nofix, and so are placed in NEITHER.

Consecutive `NLCHAR`s are treated the same as a single `NLCHAR`.

8 Concrete syntax

8.1 Introduction

The concrete syntax defines the syntax of the Z language: every sentence of the Z language is recognised by this syntax, and all sentences recognised by this syntax are sentences of the Z language. The concrete syntax is written in terms of the tokens generated by the lexis (clause 7). There are no terminal symbols within this syntax, so as to establish a formal connection with that lexis. Sequences of tokens that are not recognised by this syntax are not sentences of the Z language and are thus excluded from consideration by subsequent phases and so are not given a semantics by this International Standard.

A parser conforming to this concrete syntax converts a concrete sentence to a parse tree.

The non-terminal symbols of the concrete syntax that are written as mathematical symbols or are entirely `CAPITALIZED` or Roman are Z tokens defined in the lexis (claue 7). The other non-terminal symbols are written in `MixedCase` and are defined within the concrete syntax.

8.2 Formal definition of concrete syntax

```
Specification   = { Section }                                        (* sectioned specification *)
                | { Paragraph }                                      (* anonymous specification *)
                ;

Section         = section , NAME , parents , [ NAME , { ,-tok , NAME } ] , END ,
                      { Paragraph }                                  (* inheriting section *)
                | section , NAME , END , { Paragraph }               (* base section *)
                ;

Paragraph       = [-tok , NAME , { ,-tok , NAME } , ]-tok , END      (* given types *)
                | AX , SchemaText , END                              (* axiomatic description *)
                | SCH , NAME , SchemaText , END                      (* schema definition *)
                | GENAX , [-tok , Formals , ]-tok , SchemaText , END
                                                                     (* generic axiomatic description *)
                | GENSCH , NAME , [-tok , Formals , ]-tok , SchemaText , END
                                                                     (* generic schema definition *)
                | DeclName , == , Expression , END                   (* horizontal definition *)
                | NAME , [-tok , Formals , ]-tok , == , Expression , END
                                                                     (* generic horizontal definition *)
                | GenName , == , Expression , END                    (* generic operator definition *)
                | Freetype , { & , Freetype } , END                  (* free types *)
                | ⊢? , Predicate , END                               (* conjecture *)
                | [-tok , Formals , ]-tok , ⊢? , Predicate , END     (* generic conjecture *)
                | OperatorTemplate , END                             (* operator template *)
                ;

Freetype        = NAME , ::= , Branch , { |-tok , Branch } ;         (* free type *)

Branch          = DeclName , [ ⟪ , Expression , ⟫ ] ;                (* element or injection *)

Formals         = NAME , { ,-tok , NAME } ;                          (* generic parameters *)

Predicate       = Predicate , NL , Predicate                         (* newline conjunction *)
                | Predicate , ;-tok , Predicate                      (* semicolon conjunction *)
                | ∀ , SchemaText , • , Predicate                     (* universal quantification *)
                | ∃ , SchemaText , • , Predicate                     (* existential quantification *)
                | ∃_1 , SchemaText , • , Predicate                   (* unique existential quantification *)
                | Predicate , ⇔ , Predicate                          (* equivalence *)
                | Predicate , ⇒ , Predicate                          (* implication *)
                | Predicate , ∨ , Predicate                          (* disjunction *)
                | Predicate , ∧ , Predicate                          (* conjunction *)
                | ¬ , Predicate                                      (* negation *)
                | Relation                                           (* relation operator application *)
                | Expression                                         (* schema predicate *)
                | true                                               (* truth *)
                | false                                              (* falsity *)
                | (-tok , Predicate , )-tok                          (* parenthesized predicate *)
                ;
```



```
Expression   = ∀ , SchemaText , • , Expression           (* schema universal quantification *)
             | ∃ , SchemaText , • , Expression           (* schema existential quantification *)
             | ∃₁ , SchemaText , • , Expression          (* schema unique existential quantification *)
             | λ , SchemaText , • , Expression           (* function construction *)
             | μ , SchemaText , • , Expression           (* definite description *)
             | let , DeclName , == , Expression
                 { ;-tok , DeclName , == , Expression }
                   , • , Expression                      (* substitution expression *)
             | Expression , ⇔ , Expression               (* schema equivalence *)
             | Expression , ⇒ , Expression               (* schema implication *)
             | Expression , ∨ , Expression               (* schema disjunction *)
             | Expression , ∧ , Expression               (* schema conjunction *)
             | ¬ , Expression                            (* schema negation *)
             | if , Predicate , then , Expression , else , Expression   (* conditional *)
             | Expression , ⨾ , Expression               (* schema composition *)
             | Expression , ≫ , Expression               (* schema piping *)
             | Expression , \ , (-tok , DeclName , { ,-tok , DeclName } , )-tok
                                                         (* schema hiding *)
             | Expression , ↾ , Expression               (* schema projection *)
             | pre , Expression                          (* schema precondition *)
             | Expression , × , Expression , { × , Expression }   (* Cartesian product *)
             | ℙ , Expression                            (* powerset *)
             | Application                               (* function and generic operator application *)
             | Expression , Expression                   (* application *)
             | Expression , STROKE                       (* schema decoration *)
             | Expression , [-tok , DeclName , / , DeclName ,
                 { ,-tok , DeclName , / , DeclName } , ]-tok   (* schema renaming *)
             | Expression , . , RefName                  (* binding selection *)
             | Expression , . , NUMERAL                  (* tuple selection *)
             | θ , Expression , { STROKE }               (* binding construction *)
             | RefName                                   (* reference *)
             | RefName , [-tok , Expression , { ,-tok , Expression } , ]-tok
                                                         (* generic instantiation *)
             | NUMERAL                                   (* number literal *)

             | {-tok , [ Expression , { ,-tok , Expression } ] , }-tok   (* set extension *)
             | {-tok , SchemaText , • , Expression , }-tok   (* set comprehension *)
             | ( ( {-tok , SchemaText , }-tok )  — ( {-tok , }-tok ) )
                         — ( {-tok , Expression , }-tok )
                                                         (* characteristic set comprehension *)
             | ( [-tok , SchemaText , ]-tok )  — ( [-tok , Expression , ]-tok )
                                                         (* schema construction *)
             | ⦉ , [ DeclName , == , Expression ,
                 { ,-tok , DeclName , == , Expression } ] , ⦊   (* binding extension *)
             | (-tok , Expression , ,-tok , Expression , { ,-tok , Expression } , )-tok
                                                         (* tuple extension *)
             | (-tok , μ , SchemaText , )-tok            (* characteristic definite description *)
             | (-tok , Expression , )-tok                (* parenthesized expression *)
             ;

SchemaText   = [ DeclPart ] , [ |-tok , Predicate ] ;

DeclPart     = Declaration , { ( ;-tok | NL ) , Declaration } ;
```

```
Declaration      = DeclName , { ,-tok , DeclName } , : , Expression
                 | DeclName , == , Expression
                 | Expression
                 ;

OperatorTemplate = relation , Template
                 | function , CategoryTemplate
                 | generic , CategoryTemplate
                 ;

CategoryTemplate = Prec , PrefixTemplate
                 | Prec , PostfixTemplate
                 | Prec , Assoc , InfixTemplate
                 | NofixTemplate
                 ;

Prec             = NUMERAL ;

Assoc            = leftassoc
                 | rightassoc
                 ;

Template         = PrefixTemplate
                 | PostfixTemplate
                 | InfixTemplate
                 | NofixTemplate
                 ;

PrefixTemplate   = (-tok , PrefixName , )-tok
                 | (-tok , ℙ , _ , )-tok
                 ;

PostfixTemplate  = (-tok , PostfixName , )-tok ;

InfixTemplate    = (-tok , InfixName , )-tok ;

NofixTemplate    = (-tok , NofixName , )-tok ;

DeclName         = NAME
                 | OpName
                 ;

RefName          = NAME
                 | (-tok , OpName , )-tok
                 ;

OpName           = PrefixName
                 | PostfixName
                 | InfixName
                 | NofixName
                 ;

PrefixName       = PRE , _
                 | PREP , _
                 | L , { _ , ES | ,, , SS } , ( _ , ERE | ,, , SRE ) , _
                 | LP , { _ , ES | ,, , SS } , ( _ , EREP | ,, , SREP ) , _
                 ;
```



```
PostfixName      = _ , POST
                 | _ , POSTP
                 | _ , EL  , { _ , ES | ,, , SS } , ( _ , ER | ,, , SR )
                 | _ , ELP , { _ , ES | ,, , SS } , ( _ , ERP | ,, , SRP )
                 ;

InfixName        = _ , I , _
                 | _ , IP , _
                 | _ , EL  , { _ , ES | ,, , SS } , ( _ , ERE | ,, , SRE ) , _
                 | _ , ELP , { _ , ES | ,, , SS } , ( _ , EREP | ,, , SREP ) , _
                 ;

NofixName        = L , { _ , ES | ,, , SS } , ( _ , ER | ,, , SR )
                 | LP , { _ , ES | ,, , SS } , ( _ , ERP | ,, , SRP )
                 ;

GenName          = PrefixGenName
                 | PostfixGenName
                 | InfixGenName
                 | NofixGenName
                 ;

PrefixGenName    = PRE , NAME
                 | L , { NAME , ( ES | SS ) } , NAME , ( ERE | SRE ) , NAME
                 ;

PostfixGenName   = NAME , POST
                 | NAME , EL , { NAME , ( ES | SS ) } , NAME , ( ER | SR )
                 ;

InfixGenName     = NAME , I , NAME
                 | NAME , EL , { NAME , ( ES | SS ) } , NAME , ( ERE | SRE ) , NAME
                 ;

NofixGenName     = L , { NAME , ( ES | SS ) } , NAME , ( ER | SR ) ;

Relation         = PrefixRel
                 | PostfixRel
                 | InfixRel
                 | NofixRel
                 ;

PrefixRel        = PREP , Expression
                 | LP , ExpSep , ( Expression , EREP | ExpressionList , SREP ) , Expression
                 ;

PostfixRel       = Expression , POSTP
                 | Expression , ELP , ExpSep , ( Expression , ERP | ExpressionList , SRP )
                 ;

InfixRel         = Expression , ( ∈ | =-tok | IP ) , Expression
                       { ( ∈ | =-tok | IP ) , Expression }
                 | Expression , ELP , ExpSep ,
                     ( Expression , EREP | ExpressionList , SREP ) , Expression
                 ;

NofixRel         = LP , ExpSep , ( Expression , ERP | ExpressionList , SRP ) ;
```



```
Application      = PrefixApp
                 | PostfixApp
                 | InfixApp
                 | NofixApp
                 ;

PrefixApp        = PRE , Expression
                 | L , ExpSep , ( Expression , ERE | ExpressionList , SRE ) , Expression
                 ;

PostfixApp       = Expression , POST
                 | Expression , EL , ExpSep , ( Expression , ER | ExpressionList , SR )
                 ;

InfixApp         = Expression , I , Expression
                 | Expression , EL , ExpSep ,
                     ( Expression , ERE | ExpressionList , SRE ) , Expression
                 ;

NofixApp         = L , ExpSep , ( Expression , ER | ExpressionList , SR ) ;

ExpSep           = { Expression , ES | ExpressionList , SS } ;

ExpressionList   = [ Expression , { ,-tok , Expression } ] ;
```

8.3 Operator precedences and associativities

Table 29 defines the relative precedences of the productions of `Expression` and `Predicate`. The rows in the table are ordered so that the entries with higher precedence (and so that bind more strongly) appear nearer the top of the table than those with lower precedence (that bind more weakly). Associativity has significance only in determining the nesting of applications involving non-associative operators of the same precedence. Explicitly-defined function and generic operator applications have a range of precedences specified numerically in the corresponding operator template paragraph. Cartesian product expressions have precedence value 8 and powerset expressions have precedence value 80 within that numeric range.

8.4 Additional syntactic restrictions and notes

All prefix operators are right associative. All postfix operators are left associative. Different associativities shall not be used at the same precedence level by operator template paragraphs in the same scope.

`STROKE` is used in three contexts: within `NAME`s, in binding construction expressions, and in schema decoration expressions. The condition for a `STROKE` to be considered as part of a `NAME` was given in 7.3. Other `STROKE`s are considered to be parts of binding construction expressions if they can be when interpreted from left to right. The schema decoration expression interpretation is considered last.

EXAMPLE 1 In $\theta S' \, ' \, '$, the first $'$ is part of the `NAME` S', the second $'$ can be part of the binding construction expression, and the third $'$ is syntactically part of a schema decoration expression, though that will be rejected by the type inference rules as only schemas can be decorated, not bindings.

A predicate can be just an expression, yet the same logical operators ($\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftrightarrow$, $\forall$, $\exists$, $\exists_1$) can be used in both expressions and predicates. Where a predicate is expected, and one of these logical operators is used on expressions, there is an ambiguity: either the whole logical operation is an expression and that expression is used as a predicate, or the whole logical operation is a predicate involving expressions each used as a predicate. This ambiguity is benign, as both interpretations have the same precedence, associativity and meaning.

The `NUMERAL` in a tuple selection expression is interpreted in decimal (base ten). That the number is in the appropriate range is checked by the relevant type inference rule (13.2.6.7). Leading zeroes shall be accepted and ignored.

Table 29 – Operator precedences and associativities

Productions	Associativity
binding construction	
binding selection, tuple selection	
schema renaming	
schema decoration	
application	left
Cartesian product, powerset, function and generic operator application	
schema precondition	
schema projection	left
schema hiding	
schema piping	
schema composition	
conditional	
substitution expression	
definite description	
function construction	
relation operator application	
negation	
conjunction	
disjunction	
implication	right
equivalence	
universal, existential and unique existential quantifications	
newline conjunction, semicolon conjunction	

A section header shall be parsed in its entirety before bringing the declarations and operator templates of its parent sections into scope.

NOTE 1 This prevents surprises when the name of a parent section is the same as the name of an operator defined in another parent.

In the `Template` rule's auxiliaries, the name of each of the operator tokens shall not have any `STROKE`s.

NOTE 2 This is so that any common decoration of those words can be treated as an application of a decorated instance of that operator.

NOTE 3 The order of productions in the `Predicate` and `Expression` rules is based roughly on the precedences of their operators. Some productions have the same precedence as their neighbours, and so the separate table of operator precedences is necessary.

NOTE 4 One way of parsing nested operator applications at different user-defined levels of precedence and associativity is explained by Lalonde and des Rivieres [8]. Using distinct variants of the operator tokens `PRE`|...|`SS` for relational operators from those for function and generic operators allows that transformation to avoid dealing with those notations whose precedences lie between the relations and the functions, such as the schema operations.

NOTE 5 The juxtaposition of two expressions $e_1 e_2$ is always parsed as the application of function e_1 to argument e_2, never as the application of relation e_1 to argument e_2 which in some previous dialects of Z, e.g. King *et al* [7], was equivalent to the relation $e_2 \in e_1$. In Standard Z, membership is the normal form of all relational predicates (see 12.2.10), and juxtaposition is the normal form (canonical representation) of all application expressions (see 12.2.11).

NOTE 6 In the `PrefixTemplate` rule, the production for powerset enables explicit definition of $\mathbb{P}_1$ in the mathematical toolkit to coexist with the treatment of $\mathbb{P}$ as a keyword by the lexis.

NOTE 7 The syntax of conjectures is deliberately simple. This is so as to be compatible with the syntaxes of sequents as found in as many different theorem provers as possible, while establishing a common form to enable interchange.

NOTE 8 Implementations of parsers commonly inspect the next token from the input stream and have to decide there and then whether that token is another token in an incomplete phrase or whether the current phrase is complete and the token is starting a new phrase. The tokens defined by the lexis (clause 7) are insufficient for such an implementation of a parser.

EXAMPLE 2 At the first comma in the set extension $\{x, y, z\}$ the x shall be seen to be an expression, yet the phrase might yet turn out to be the set comprehension $\{x, y, z : e\}$.

EXAMPLE 3 At the opening square bracket in the application to a schema construction $i\ [e_1;\ e_2]$ the name i shall be seen to be an expression, yet the phrase might yet turn out to be the generic instantiation $i[e_1]$.

One solution to such problems is to try all possible parses and accept the one that works. Another solution is to have the lexer lookahead far enough to be able to distinguish which of the alternative cases is present, and to provide the parser with one of several distinct tokens. Expressions $\{x, y, z\}$ and $\{x, y, z : e\}$ can be distinguished by looking ahead from a comma, over following alternating names (including operator names) and commas, for a : or == token, and using distinct comma tokens depending on whether that is seen or not. Expressions $i\ [e_1;\ e_2]$ and $i[e_1]$ can be distinguished by looking ahead from the open square bracket for the matching closing square bracket, stopping if a ; -tok, :, == or |-tok token is encountered, and using distinct open square bracket tokens for the matched and stopped cases.

NOTE 9 An `ExpressionList` phrase will be regarded as an expression whose value is a sequence (see 12.2.12). In defining operators that take such `ExpressionList`s as arguments, it is convenient to have the operations of *sequence_toolkit* (see B.8) in scope.

NOTE 10 The `ExpressionList` rule is used only in operator applications, not in set extension, tuple extension and generic instantiation expressions, so that syntactic transformation 12.2.12 is applied only to operator applications.

8.5 Box renderings

There are two different sets of box renderings in widespread use, as illustrated here. Any particular presentation of a section shall use one set or the other throughout. The middle line shall be omitted when the paragraph has no predicates, but otherwise shall be retained if the paragraph has no declarations. The outlines need be only as wide as the text, but are here shown as wide as the page.

8.5.1 First box rendering

The following four paragraphs illustrate the first of two alternative renderings of box tokens.

An axiomatic paragraph, involving the `AX`, |-tok and `END` tokens, shall have this box rendering.

```
│ DeclPart
├──────────
│ Predicate
```

A schema paragraph, involving the `SCH`, |-tok and `END` tokens, shall have this box rendering.

```
┌─NAME──────────────────────────────┐
│ DeclPart
├──────────
│ Predicate
└───────────────────────────────────┘
```

A generic axiomatic paragraph, involving the `GENAX`, |-tok and `END` tokens, shall have this box rendering.

```
╒═[Formals]═════════════════════════
│ DeclPart
├──────────
│ Predicate
```


A generic schema paragraph, involving the `GENSCH`, |-tok and `END` tokens, shall have this box rendering.

```
NAME [Formals]
DeclPart
Predicate
```

8.5.2 Second box rendering

The following four paragraphs illustrate the second of two alternative renderings of box tokens.

An axiomatic paragraph, involving the `AX`, |-tok and `END` tokens, shall have this box rendering.

```
DeclPart
Predicate
```

A schema paragraph, involving the `SCH`, |-tok and `END` tokens, shall have this box rendering.

```
NAME
DeclPart
Predicate
```

A generic axiomatic paragraph, involving the `GENAX`, |-tok and `END` tokens, shall have this box rendering.

```
[Formals]
DeclPart
Predicate
```

A generic schema paragraph, involving the `GENSCH`, |-tok and `END` tokens, shall have this box rendering.

```
NAME [Formals]
DeclPart
Predicate
```

9 Characterisation rules

9.1 Introduction

The characterisation rules together map the parse tree of a concrete syntax sentence to the parse tree of an equivalent concrete syntax sentence in which all implicit characteristic tuples have been made explicit.

Only concrete trees that are mapped to different trees are given explicit characterisation rules. The characterisation rules are listed in the same order as the corresponding productions of the concrete syntax.

Characteristic tuples are calculated from schema texts by the metalanguage function *chartuple* (9.2).

9.2 Characteristic tuple

A characteristic tuple is computed in two phases: *charac*, which returns a sequence of expressions, and *mktuple*, which converts that sequence into the characteristic tuple.

$$chartuple\ t \quad = \quad mktuple\ (charac\ t)$$

Sequences of expressions are enclosed between metalanguage brackets $\langle$ and $\rangle$, in general $\langle e_1, ..., e_n \rangle$. Two sequences of expressions are concatenated by the $\frown$ operator.

$$\langle e_1, ..., e_n \rangle \frown \langle e_{n+1}, ..., e_{n+m} \rangle \quad = \quad \langle e_1, ..., e_n, e_{n+1}, ..., e_{n+m} \rangle$$

Unlike the mathematical metalanguage of 4.2, the operands of these two notations are not semantic values but parse trees of the concrete syntax.

$$\begin{array}{rcll}
charac\ (de_1;\ ...;\ de_n \mid p) & = & charac\ (de_1;\ ...;\ de_n) & \\
charac\ (de_1;\ ...;\ de_n) & = & charac\ de_1 \frown ... \frown charac\ de_n & \quad where\ n \geq 1 \\
charac\ () & = & \langle \lblot\ \rblot \rangle & \\
charac\ (i_1, ..., i_n : e) & = & \langle i_1, ..., i_n \rangle & \\
charac\ (i == e) & = & \langle i \rangle & \\
charac\ e\ ^* & = & \langle \theta\ e^* \rangle &
\end{array}$$

$$\begin{array}{rcll}
mktuple\ \langle e \rangle & = & e & \\
mktuple\ \langle e_1, ..., e_n \rangle & = & (e_1, ..., e_n) & \quad where\ n \geq 2
\end{array}$$

NOTE 1 In the last case of *charac*, the type inference rule 13.2.6.9 ensures that e is a schema.

NOTE 2 In *mktuple*, the result is a Z expression, so the brackets in its second equation are those of a tuple extension.

9.3 Formal definition of characterisation rules

9.3.1 Function construction expression

The value of the function construction expression $\lambda\ t \bullet e$ is the function associating values of the characteristic tuple of t with corresponding values of e.

$$\lambda\ t \bullet e \quad \Longrightarrow \quad \{t \bullet (chartuple\ t, e)\}$$

It is semantically equivalent to the set of pairs representation of that function.

9.3.2 Characteristic set comprehension expression

The value of the characteristic set comprehension expression $\{t\}$ is the set of the values of the characteristic tuple of t.

$$\{t\} \quad \Longrightarrow \quad \{t \bullet chartuple\ t\}$$

It is semantically equivalent to the corresponding set comprehension expression in which the characteristic tuple is made explicit.

9.3.3 Characteristic definite description expression

The value of the characteristic definite description expression $(\mu\ t)$ is the sole value of the characteristic tuple of schema text t.

$$(\mu\ t) \quad \Longrightarrow \quad \mu\ t \bullet chartuple\ t$$

It is semantically equivalent to the corresponding definite description expression in which the characteristic tuple is made explicit.

10 Annotated syntax

10.1 Introduction

The annotated syntax defines a language that includes all sentences that could be produced by application of the syntactic transformation rules (clause 12) to sentences of the concrete syntax (clause 8). This language's set of sentences would be a subset of that defined by the concrete syntax but for introduction of type annotations and use of expressions in place of schema texts.

Like the concrete syntax, this annotated syntax is written in terms of the tokens generated by the lexis; there are no terminal symbols within this syntax. Three additional tokens are used besides those defined in the lexis: `GIVEN`, `GENTYPE`, and ⨾. An additional character, ⋈, in included in the *STROKECHAR* and *WORDGLUE* classes. It is assumed that ⋈ is distinct from the characters used in concrete phrases. This restriction ensures that its use in forming single names from the constituent words of operators produces results that cannot clash with any other names. It also ensures that its use as a stroke on inclusion declarations cannot result in any captures of references to other declarations. Further characters ♠ and ♡ are included in the `STROKECHAR` class. They are also assumed to be distinct from the characters used in concrete phrases. They are used in defining the semantics of types, again for the purpose of avoiding variable captures.

There are no parentheses in the annotated syntax as defined here. A sentence or phrase of the annotated syntax should be thought of as a tree structure of nested formulae. When presented as linear text, however, the precedences of the concrete syntax may be assumed and parentheses may be inserted to override those precedences. The precedence of the type annotation ⨾ operator is then weaker than all other operators, and the precedences and associativities of the type notations are analogous to those of the concrete notations of similar appearance.

NOTE 1 This annotated syntax permits some verification of the syntactic transformation rules to be performed.

NOTE 2 The annotated syntax is similar to an abstract syntax used in a tool, but the level of abstraction effected by the characterization rules and syntactic transformation rules might not be appropriate for a tool.

10.2 Formal definition of annotated syntax

```
Specification   = { Section } ;                                     (* sectioned specification *)

Section         = section , NAME , parents , [ NAME , { ,-tok , NAME } ] , END ,
                      { Paragraph } , [ ⨾ , SectTypeEnv ] ;                (* inheriting section *)

Paragraph       = [-tok , NAME , { ,-tok , NAME } , ]-tok , END ,
                      [ ⨾ , Signature ]                                        (* given types *)
                | AX , Expression , END ,
                      [ ⨾ , Signature ]                               (* axiomatic description *)
                | GENAX , [-tok , NAME , { ,-tok , NAME } , ]-tok , Expression , END ,
                      [ ⨾ , Signature ]                       (* generic axiomatic description *)
                | NAME , ::= , NAME , [ ⟪ , Expression , ⟫ ] ,
                    { |-tok , NAME , [ ⟪ , Expression , ⟫ ] } ,
                    { & , NAME , ::= , NAME , [ ⟪ , Expression , ⟫ ] ,
                    { |-tok , NAME , [ ⟪ , Expression , ⟫ ] } } , END ,
                      [ ⨾ , Signature ]                                         (* free types *)
                | ⊢? , Predicate , END ,
                      [ ⨾ , Signature ]                                         (* conjecture *)
                | [-tok , NAME , { ,-tok , NAME } , ]-tok , ⊢? , Predicate , END ,
                      [ ⨾ , Signature ]                                 (* generic conjecture *)
                ;
```



```
Predicate      = Expression , ∈ , Expression                (* membership *)
               | true                                       (* truth *)
               | ¬ , Predicate                              (* negation *)
               | Predicate , ∧ , Predicate                  (* conjunction *)
               | ∀ , Expression , • , Predicate             (* universal quantification *)
               | ∃₁ , Expression , • , Predicate            (* unique existential quantification *)
               ;
```

```
Expression       = NAME ,
                       [ ⦂ , Type ]                                        (* reference *)
                 | NAME , [-tok , Expression , { ,-tok , Expression } , ]-tok ,
                       [ ⦂ , Type ]                            (* generic instantiation *)
                 | {-tok , [ Expression , { ,-tok , Expression } ] , }-tok ,
                       [ ⦂ , Type ]                                    (* set extension *)
                 | {-tok , Expression , • , Expression , }-tok ,
                       [ ⦂ , Type ]                                (* set comprehension *)
                 | ℙ , Expression ,
                       [ ⦂ , Type ]                                         (* powerset *)
                 | (-tok , Expression , ,-tok , Expression , { ,-tok , Expression } , )-tok ,
                       [ ⦂ , Type ]                                  (* tuple extension *)
                 | Expression , . , NUMERAL ,
                       [ ⦂ , Type ]                                  (* tuple selection *)
                 | ⦉ , NAME , == , Expression ,
                       { ,-tok , NAME , == , Expression } , ⦊ ,
                           [ ⦂ , Type ]                            (* binding extension *)
                 | θ , Expression , { STROKE } ,
                       [ ⦂ , Type ]                             (* binding construction *)
                 | Expression , . , NAME ,
                       [ ⦂ , Type ]                                (* binding selection *)
                 | Expression , Expression ,
                       [ ⦂ , Type ]                                      (* application *)
                 | μ , Expression , • , Expression ,
                       [ ⦂ , Type ]                             (* definite description *)
                 | [-tok , NAME , : , Expression , ]-tok ,
                       [ ⦂ , Type ]                            (* variable construction *)
                 | [-tok , Expression , |-tok , Predicate , ]-tok ,
                       [ ⦂ , Type ]                              (* schema construction *)
                 | ¬ , Expression ,
                       [ ⦂ , Type ]                                  (* schema negation *)
                 | Expression , ∧ , Expression ,
                       [ ⦂ , Type ]                               (* schema conjunction *)
                 | Expression , \ , (-tok , NAME , { ,-tok , NAME } , )-tok ,
                       [ ⦂ , Type ]                                    (* schema hiding *)
                 | ∀ , Expression , • , Expression ,
                       [ ⦂ , Type ]                (* schema universal quantification *)
                 | ∃₁ , Expression , • , Expression ,
                       [ ⦂ , Type ]       (* schema unique existential quantification *)
                 | Expression , [-tok , NAME , / , NAME ,
                       { ,-tok , NAME , / , NAME } , ]-tok ,
                           [ ⦂ , Type ]                             (* schema renaming *)
                 | pre , Expression ,
                       [ ⦂ , Type ]                              (* schema precondition *)
                 | Expression , ⨾ , Expression ,
                       [ ⦂ , Type ]                               (* schema composition *)
                 | Expression , ≫ , Expression ,
                       [ ⦂ , Type ]                                    (* schema piping *)
                 | Expression , STROKE ,
                       [ ⦂ , Type ]                                (* schema decoration *)
                 ;
```



```
SectTypeEnv    = [ NAME , : , (-tok , NAME , ,-tok , Type , )-tok ,
                   { ;-tok , NAME , : , (-tok , NAME , ,-tok , Type , )-tok } ] ;

Type           = GIVEN , NAME                                    (* given type *)
               | GENTYPE , NAME                      (* generic parameter type *)
               | ℙ , Type                                     (* powerset type *)
               | Type , × , Type , { × , Type }      (* Cartesian product type *)
               | [-tok , Signature , ]-tok                      (* schema type *)
               | [-tok , NAME , { ,-tok , NAME } , ]-tok ,
                     Type , [ ,-tok , Type ]                   (* generic type *)
               | α , { STROKE }                               (* variable type *)
               ;

Signature      = [ NAME , : , Type , { ;-tok , NAME , : , Type } ]
               | β , { STROKE }                          (* variable signature *)
               | ϵ                                          (* empty signature *)
               ;
```

10.3 Notes

NOTE 1 More free types than necessary are permitted by this syntax: as a result of syntactic transformation 12.2.3.5, all elements appear before all injections.

NOTE 2 More types than necessary are permitted by this syntax: generic type notation is used at only the outermost level of a type.

11 Prelude

11.1 Introduction

The prelude is a Z section. It is an implicit parent of every other section. It assists in defining the meaning of number literal expressions (see 12.2.6.9), and the list arguments of operators (see 12.2.12), via syntactic transformation rules later in this International Standard. The prelude is presented here using the mathematical lexis.

11.2 Formal definition of prelude

section *prelude*

The section is called *prelude* and has no parents.

generic 80 ($\mathbb{P}\_$)

The precedence of the prefix generic operator $\mathbb{P}$ is 80.

$[\mathbb{A}]$

The given type $\mathbb{A}$, pronounced "arithmos", provides a supply of values for use in specifying number systems.

| $\mathbb{N} : \mathbb{P}\,\mathbb{A}$

The set of natural numbers, $\mathbb{N}$, is a subset of $\mathbb{A}$.

| $number\_literal\_0 : \mathbb{N}$
| $number\_literal\_1 : \mathbb{N}$

0 and 1 are natural numbers, all uses of which are transformed to references to these declarations (see 12.2.6.9).

function 30 leftassoc ($\_ + \_$)

$$
\begin{array}{|l}
\_ + \_ : \mathbb{P}\,((\mathbb{A} \times \mathbb{A}) \times \mathbb{A}) \\
\hline
\forall\, m, n : \mathbb{N} \bullet \exists_1\, p : (\_ + \_) \bullet p.1 = (m, n) \\
\forall\, m, n : \mathbb{N} \bullet m + n \in \mathbb{N} \\
\forall\, m, n : \mathbb{N} \mid m + 1 = n + 1 \bullet m = n \\
\forall\, m : \mathbb{N} \bullet \neg\, m + 1 = 0 \\
\forall\, w : \mathbb{P}\ \mathbb{N} \mid 0 \in w \land (\forall\, y : w \bullet y + 1 \in w) \bullet w = \mathbb{N} \\
\forall\, m : \mathbb{N} \bullet m + 0 = m \\
\forall\, m, n : \mathbb{N} \bullet m + (n + 1) = m + n + 1
\end{array}
$$

Addition is defined here for natural numbers. (It is extended to integers in the mathematical toolkit, annex B.) Addition is a function. The sum of two natural numbers is a natural number. The operation of adding 1 is an injection on natural numbers, and produces a result different from 0. There is an induction constraint that all natural numbers are either 0 or are formed by adding 1 to another natural number. 0 is an identity of addition. Addition is associative.

NOTE The definition of addition is equivalent to the following definition, which is written using notation from the mathematical toolkit (and so is unsuitable as the normative definition here).

$$
\begin{array}{|l}
\_ + \_ : \mathbb{A} \times \mathbb{A} \leftrightarrow \mathbb{A} \\
\hline
(\mathbb{N} \times \mathbb{N}) \lhd (\_ + \_) \in (\mathbb{N} \times \mathbb{N}) \rightarrow \mathbb{N} \\
\lambda\, n : \mathbb{N} \bullet n + 1 \in \mathbb{N} \rightarrowtail \mathbb{N} \\
\mathit{disjoint} \langle \{0\}, \{n : \mathbb{N} \bullet n + 1\} \rangle \\
\forall\, w : \mathbb{P}\mathbb{N} \mid \{0\} \cup \{n : w \bullet n + 1\} \subseteq w \bullet w = \mathbb{N} \\
\forall\, m : \mathbb{N} \bullet m + 0 = m \\
\forall\, m, n : \mathbb{N} \bullet m + (n + 1) = m + n + 1
\end{array}
$$

12 Syntactic transformation rules

12.1 Introduction

The syntactic transformation rules together map the parse tree of a concrete syntax sentence to the parse tree of a semantically equivalent annotated syntax sentence. The resulting annotated parse trees may refer to definitions of the prelude.

Although exhaustive application of the syntactic transformation rules produces annotated parse trees, individual syntactic transformation rules can produce a mixture of concrete and annotated notation. Explicit distinction of the two is not done, as it would be cumbersome and detract from readability.

Only concrete trees that are not in the annotated syntax are given explicit syntactic transformation rules. The syntactic transformation rules are listed in the same order as the corresponding productions of the concrete syntax. Where an individual concrete syntax production is expressed using alternations, a separate syntactic transformation rule is given for each alternative.

All applications of syntactic transformation rules that generate new declarations shall choose the names of those declarations to be such that they do not capture references during subsequent typechecking.

Rules that generate type annotations generate annotations with fresh variables each time they are applied.

12.2 Formal definition of syntactic transformation rules

12.2.1 Specification

12.2.1.1 Anonymous specification

The anonymous specification $d_1 \ldots d_n$ is semantically equivalent to the sectioned specification comprising a single section containing those paragraphs with the mathematical toolkit of annex B as its parent.

$$d_1 \ldots d_n \implies \mathit{Mathematical\ toolkit}\ \text{section}\ \mathit{Specification}\ \text{parents}\ \mathit{standard\_toolkit}\ \texttt{END}\ d_1 \ldots d_n$$

In this transformation, *Mathematical toolkit* denotes the entire text of annex B. The name given to the section is not important: it need not be *Specification*, though it may not be *prelude* or that of any section of the mathematical toolkit.

12.2.2 Section

12.2.2.1 Base section

The base section section i END d_1 ... d_n is semantically equivalent to the inheriting section that inherits from no parents (bar *prelude*).

$$\text{section}\ \ i\ \mathtt{END}\ d_1\ ...\ d_n \quad \Longrightarrow \quad \text{section}\ \ i\ \ \text{parents}\ \mathtt{END}\ d_1\ ...\ d_n$$

12.2.3 Paragraph

12.2.3.1 Schema definition paragraph

The schema definition paragraph SCH i t END introduces the global name i, associating it with the schema that is the value of t.

$$\mathtt{SCH}\ i\ t\ \mathtt{END} \quad \Longrightarrow \quad \mathtt{AX}\ [i == t]\ \mathtt{END}$$

The paragraph is semantically equivalent to the axiomatic description paragraph whose sole declaration associates the schema's name with the expression resulting from syntactic transformation of the schema text.

12.2.3.2 Generic schema definition paragraph

The generic schema definition paragraph GENSCH i $[i_1, ..., i_n]$ t END can be instantiated to produce a schema definition paragraph.

$$\mathtt{GENSCH}\ i\ [i_1, ..., i_n]\ t\ \mathtt{END} \quad \Longrightarrow \quad \mathtt{GENAX}\ [i_1, ..., i_n]\ [i == t]\ \mathtt{END}$$

It is semantically equivalent to the generic axiomatic description paragraph with the same generic parameters and whose sole declaration associates the schema's name with the expression resulting from syntactic transformation of the schema text.

12.2.3.3 Horizontal definition paragraph

The horizontal definition paragraph $i == e$ END introduces the global name i, associating with it the value of e.

$$i == e\ \mathtt{END} \quad \Longrightarrow \quad \mathtt{AX}\ [i == e]\ \mathtt{END}$$

It is semantically equivalent to the axiomatic description paragraph that introduces the same single declaration.

12.2.3.4 Generic horizontal definition paragraph

The generic horizontal definition paragraph i $[i_1, ..., i_n] == e$ END can be instantiated to produce a horizontal definition paragraph.

$$i\ [i_1, ..., i_n] == e\ \mathtt{END} \quad \Longrightarrow \quad \mathtt{GENAX}\ [i_1, ..., i_n]\ [i == e]\ \mathtt{END}$$

It is semantically equivalent to the generic axiomatic description paragraph with the same generic parameters and that introduces the same single declaration.

12.2.3.5 Free types paragraph

The transformation of free types paragraphs is done in two stages. First, the branches are permuted to bring elements to the front and injections to the rear.

$$...\mid g\langle\!\langle e\rangle\!\rangle \mid h \mid ... \quad \Longrightarrow \quad ...\mid h \mid g\langle\!\langle e\rangle\!\rangle \mid ...$$

Exhaustive application of this syntactic transformation rule effects a sort.

The second stage requires implicit generic instantiations to have been filled in, which is done during typechecking (section 13.2.3.3). Hence that second stage is delayed until after typechecking, where it appears in the form of a semantic transformation rule (section 14.2.3.1).

12.2.4 Operator template

There are no syntactic transformation rules for operator template paragraphs; rather, operator template paragraphs determine which syntactic transformation rule to use for each phrase that refers to or applies an operator.

12.2.5 Predicate

12.2.5.1 Newline conjunction predicate

The newline conjunction predicate $p_1 \; \mathtt{NL} \; p_2$ is *true* if and only if both its predicates are *true*.

$$p_1 \; \mathtt{NL} \; p_2 \quad \Longrightarrow \quad p_1 \land p_2$$

It is semantically equivalent to the conjunction predicate $p_1 \land p_2$.

12.2.5.2 Semicolon conjunction predicate

The semicolon conjunction predicate $p_1 ; \; p_2$ is *true* if and only if both its predicates are *true*.

$$p_1 ; \; p_2 \quad \Longrightarrow \quad p_1 \land p_2$$

It is semantically equivalent to the conjunction predicate $p_1 \land p_2$.

12.2.5.3 Existential quantification predicate

The existential quantification predicate $\exists \; t \bullet p$ is *true* if and only if p is *true* for at least one value of t.

$$\exists \; t \bullet p \quad \Longrightarrow \quad \neg \; \forall \; t \bullet \neg \; p$$

It is semantically equivalent to p being *false* for not all values of t.

12.2.5.4 Equivalence predicate

The equivalence predicate $p_1 \Leftrightarrow p_2$ is *true* if and only if both p_1 and p_2 are *true* or neither is *true*.

$$p_1 \Leftrightarrow p_2 \quad \Longrightarrow \quad (p_1 \Rightarrow p_2) \land (p_2 \Rightarrow p_1)$$

It is semantically equivalent to each of p_1 and p_2 being *true* if the other is *true*.

12.2.5.5 Implication predicate

The implication predicate $p_1 \Rightarrow p_2$ is *true* if and only if p_2 is *true* if p_1 is *true*.

$$p_1 \Rightarrow p_2 \quad \Longrightarrow \quad \neg \; p_1 \lor p_2$$

It is semantically equivalent to p_1 being *false* disjoined with p_2 being *true*.

12.2.5.6 Disjunction predicate

The disjunction predicate $p_1 \lor p_2$ is *true* if and only if at least one of p_1 and p_2 is *true*.

$$p_1 \lor p_2 \quad \Longrightarrow \quad \neg \; (\neg \; p_1 \land \neg \; p_2)$$

It is semantically equivalent to not both of p_1 and p_2 being *false*.

12.2.5.7 Schema predicate

The schema predicate e is *true* if and only if the binding of the names in the signature of schema e satisfies the constraints of that schema.

$$e \quad \Longrightarrow \quad \theta \; e \in e$$

It is semantically equivalent to the binding constructed by $\theta \; e$ being a member of the set denoted by schema e.

12.2.5.8 Falsity predicate

The falsity predicate false is never *true*.

$$\text{false} \implies \neg\ \text{true}$$

It is semantically equivalent to the negation of true.

12.2.5.9 Parenthesized predicate

The parenthesized predicate (p) is *true* if and only if p is *true*.

$$(p) \implies p$$

It is semantically equivalent to p.

12.2.6 Expression

12.2.6.1 Schema existential quantification expression

The value of the schema existential quantification expression $\exists\ t \bullet e$ is the set of bindings of schema e restricted to exclude names that are in the signature of t, for at least one binding of the schema t.

$$\exists\ t \bullet e \implies \neg\ \forall\ t \bullet \neg\ e$$

It is semantically equivalent to the result of applying de Morgan's law.

12.2.6.2 Substitution expression

The value of the substitution expression $\text{let}\, i_1 == e_1;\ ...;\ i_n == e_n \bullet e$ is the value of e when all of its references to the names have been substituted by the values of the corresponding expressions.

$$\text{let}\, i_1 == e_1;\ ...;\ i_n == e_n \bullet e \implies \mu\ i_1 == e_1;\ ...;\ i_n == i_n \bullet e$$

It is semantically equivalent to the similar definite description expression.

12.2.6.3 Schema equivalence expression

The value of the schema equivalence expression $e_1 \Leftrightarrow e_2$ is that schema whose signature is the union of those of schemas e_1 and e_2, and whose bindings are those whose relevant restrictions are either both or neither in e_1 and e_2.

$$e_1 \Leftrightarrow e_2 \implies (e_1 \Rightarrow e_2) \land (e_2 \Rightarrow e_1)$$

It is semantically equivalent to the schema conjunction shown above.

12.2.6.4 Schema implication expression

The value of the schema implication expression $e_1 \Rightarrow e_2$ is that schema whose signature is the union of those of schemas e_1 and e_2, and whose bindings are those whose restriction to the signature of e_2 is in the value of e_2 if its restriction to the signature of e_1 is in the value of e_1.

$$e_1 \Rightarrow e_2 \implies \neg\ e_1 \lor e_2$$

It is semantically equivalent to the schema disjunction shown above.

12.2.6.5 Schema disjunction expression

The value of the schema disjunction expression $e_1 \lor e_2$ is that schema whose signature is the union of those of schemas e_1 and e_2, and whose bindings are those whose restriction to the signature of e_1 is in the value of e_1 or its restriction to the signature of e_2 is in the value of e_2.

$$e_1 \lor e_2 \implies \neg\ (\neg\ e_1 \land \neg\ e_2)$$

It is semantically equivalent to the schema negation shown above.

12.2.6.6 Conditional expression

The value of the conditional expression $\text{if } p \text{ then } e_1 \text{ else } e_2$ is the value of e_1 if p is *true*, and is the value of e_2 if p is *false*.

$$\text{if } p \text{ then } e_1 \text{ else } e_2 \quad \Longrightarrow \quad \mu\, i : \{e_1, e_2\} \mid p \land i = e_1 \lor \neg\, p \land i = e_2 \bullet i$$

It is semantically equivalent to the definite description expression whose value is either that of e_1 or that of e_2 such that if p is *true* then it is e_1 or if p is *false* then it is e_2.

12.2.6.7 Schema projection expression

The value of the schema projection expression $e_1 \upharpoonright e_2$ is the schema that is like the conjunction $e_1 \land e_2$ but whose signature is restricted to just that of schema e_2.

$$e_1 \upharpoonright e_2 \quad \Longrightarrow \quad \{e_1;\ e_2 \bullet \theta\, e_2\}$$

It is semantically equivalent to that set of bindings of names in the signature of e_2 to values that satisfy the constraints of both e_1 and e_2.

12.2.6.8 Cartesian product expression

The value of the Cartesian product expression $e_1 \times ... \times e_n$ is the set of all tuples whose components are members of the corresponding sets that are the values of its expressions.

$$e_1 \times ... \times e_n \quad \Longrightarrow \quad \{i_1 : e_1;\ ...;\ i_n : e_n \bullet (i_1, ..., i_n)\}$$

It is semantically equivalent to the set comprehension expression that declares members of the sets and assembles those members into tuples.

12.2.6.9 Number literal expression

The value of the multiple-digit number literal expression bc is the number that it denotes.

$$\begin{array}{lll} bc & \Longrightarrow & b + b + b + b + b + \\ & & b + b + b + b + b + c \end{array}$$

It is semantically equivalent to the sum of ten repetitions of the number literal expression b formed from all but the last digit, added to that last digit.

$$\begin{array}{lll} 0 & \Longrightarrow & \mathit{number\_literal\_0} \\ 1 & \Longrightarrow & \mathit{number\_literal\_1} \\ 2 & \Longrightarrow & 1 + 1 \\ 3 & \Longrightarrow & 2 + 1 \\ 4 & \Longrightarrow & 3 + 1 \\ 5 & \Longrightarrow & 4 + 1 \\ 6 & \Longrightarrow & 5 + 1 \\ 7 & \Longrightarrow & 6 + 1 \\ 8 & \Longrightarrow & 7 + 1 \\ 9 & \Longrightarrow & 8 + 1 \end{array}$$

The number literal expressions 0 and 1 are semantically equivalent to *number_literal_0* and *number_literal_1* respectively as defined in section *prelude*. The remaining digits are defined as being successors of their predecessors, using the function + as defined in section *prelude*.

NOTE These syntactic transformations are applied only to `NUMERAL` tokens that form number literal expressions, not to other `NUMERAL` tokens (those in tuple selection expressions and operator template paragraphs), as those other occurrences of `NUMERAL` do not have semantic values associated with them.

12.2.6.10 Schema construction expression

The value of the schema construction expression $[t]$ is that schema whose signature is the names declared by the schema text t, and whose bindings are those that satisfy the constraints in t.

$$[t] \quad \Longrightarrow \quad t$$

It is semantically equivalent to the schema resulting from syntactic transformation of the schema text t.

12.2.6.11 Parenthesized expression

The value of the parenthesized expression (e) is the value of expression e.

$$(e) \quad \Longrightarrow \quad e$$

It is semantically equivalent to e.

12.2.7 Schema text

There is no separate schema text class in the annotated syntax: all concrete schema texts are transformed to expressions.

12.2.7.1 `Declaration`

Each declaration is transformed to an equivalent expression.

A constant declaration is equivalent to a variable declaration in which the variable ranges over a singleton set.

$$i == e \quad \Longrightarrow \quad i : \{e\}$$

A comma-separated multiple declaration is equivalent to the conjunction of variable construction expressions in which all variables are constrained to be of the same type.

$$i_1, ..., i_n : e \quad \Longrightarrow \quad [i_1 : e \ ⨾ \ \tau_1] \wedge ... \wedge [i_n : e \ ⨾ \ \tau_1]$$

12.2.7.2 `DeclPart`

Each declaration part is transformed to an equivalent expression.

$$de_1; \ ...; \ de_n \quad \Longrightarrow \quad de_1 \wedge ... \wedge de_n$$

If `NL` tokens have been used in place of any ; s, the same transformation to $\wedge$ applies.

12.2.7.3 `SchemaText`

Given the above transformations of `Declaration` and `DeclPart`, any `DeclPart` in a `SchemaText` can be assumed to be a single expression.

A `SchemaText` with non-empty `DeclPart` and `Predicate` is equivalent to the schema construction expression containing that schema text.

$$e \mid p \quad \Longrightarrow \quad [e \mid p]$$

If both `DeclPart` and `Predicate` are omitted, the schema text is equivalent to the set containing the empty binding.

$$\Longrightarrow \quad \{\langle\!|\ |\!\rangle\}$$

If just the `DeclPart` is omitted, the schema text is equivalent to the schema construction expression in which there is a set containing the empty binding.

$$\mid p \quad \Longrightarrow \quad [\{\langle\!|\ |\!\rangle\} \mid p]$$

12.2.8 Name

These syntactic transformation rules address the concrete syntax productions `DeclName`, `RefName`, and `OpName`.

All operator names are transformed to `NAME`s, by removing spaces and replacing each _ by a Z character that is not acceptable in concrete `NAME`s. The Z character ⋈ is used for this purpose here. The resulting name is given the same `STROKE`s as the component names of the operator, all of which shall have the same `STROKE`s.

Each resulting `NAME` should be one for which there is an operator template paragraph in scope.

NOTE This excludes names made up of words from different operator templates.

EXAMPLE Given the operator templates

generic 30 leftassoc ($\_\ a\ \_\ b\ \_$)
generic 40 leftassoc ($\_\ c\ \_\ d\ \_$)

the following declaration conforms to the syntax but is excluded by this restriction.

$$X\ \ a\ \ Y\ \ d\ \ Z == X \times Y \times Z$$

In every operator name generated by syntactic transformation, for every '↘' `WORDGLUE` character in its `WORD` part, there shall be a paired following '↖' `WORDGLUE` character, for every '↗' `WORDGLUE` character in its `WORD` part, there shall be a paired following '↙' `WORDGLUE` character, and these shall occur only in nested pairs.

12.2.8.1 `PrefixName`

$$\begin{array}{rcl}
pre\ _ & \Longrightarrow & pre\bowtie \\
prep\ _ & \Longrightarrow & prep\bowtie \\
ln\ _\ ess_1\ \ldots\ _\ ess_{n-2}\ _\ ere\ _ & \Longrightarrow & ln\bowtie ess_1 \ldots \bowtie ess_{n-2}\bowtie ere\bowtie \\
ln\ _\ ess_1\ \ldots\ _\ ess_{n-2}\ _\ sre\ _ & \Longrightarrow & ln\bowtie ess_1 \ldots \bowtie ess_{n-2}\bowtie sre\bowtie \\
lp\ _\ ess_1\ \ldots\ _\ ess_{n-2}\ _\ erep\ _ & \Longrightarrow & lp\bowtie ess_1 \ldots \bowtie ess_{n-2}\bowtie erep\bowtie \\
lp\ _\ ess_1\ \ldots\ _\ ess_{n-2}\ _\ srep\ _ & \Longrightarrow & lp\bowtie ess_1 \ldots \bowtie ess_{n-2}\bowtie srep\bowtie
\end{array}$$

12.2.8.2 `PostfixName`

$$\begin{array}{rcl}
_\ post & \Longrightarrow & \bowtie post \\
_\ postp & \Longrightarrow & \bowtie postp \\
_\ el\ _\ ess_2\ \ldots\ _\ ess_{n-1}\ _\ er & \Longrightarrow & \bowtie el\bowtie ess_2 \ldots \bowtie ess_{n-1}\bowtie er \\
_\ el\ _\ ess_2\ \ldots\ _\ ess_{n-1}\ _\ sr & \Longrightarrow & \bowtie el\bowtie ess_2 \ldots \bowtie ess_{n-1}\bowtie sr \\
_\ elp\ _\ ess_2\ \ldots\ _\ ess_{n-1}\ _\ erp & \Longrightarrow & \bowtie elp\bowtie ess_2 \ldots \bowtie ess_{n-1}\bowtie erp \\
_\ elp\ _\ ess_2\ \ldots\ _\ ess_{n-1}\ _\ srp & \Longrightarrow & \bowtie elp\bowtie ess_2 \ldots \bowtie ess_{n-1}\bowtie srp
\end{array}$$

12.2.8.3 `InfixName`

$$\begin{array}{rcl}
_\ in\ _ & \Longrightarrow & \bowtie in\bowtie \\
_\ ip\ _ & \Longrightarrow & \bowtie ip\bowtie
\end{array}$$

$$
\begin{array}{rcl}
\_\, el \,\_\, ess_2 \;\ldots\; \_\, ess_{n-2} \,\_\, ere \,\_ & \Longrightarrow & \bowtie el \bowtie ess_2 \ldots \bowtie ess_{n-2} \bowtie ere \bowtie \\
\_\, el \,\_\, ess_2 \;\ldots\; \_\, ess_{n-2} \,\_\, sre \,\_ & \Longrightarrow & \bowtie el \bowtie ess_2 \ldots \bowtie ess_{n-2} \bowtie sre \bowtie \\
\_\, elp \,\_\, ess_2 \;\ldots\; \_\, ess_{n-2} \,\_\, erep \,\_ & \Longrightarrow & \bowtie elp \bowtie ess_2 \ldots \bowtie ess_{n-2} \bowtie erep \bowtie \\
\_\, elp \,\_\, ess_2 \;\ldots\; \_\, ess_{n-2} \,\_\, srep \,\_ & \Longrightarrow & \bowtie elp \bowtie ess_2 \ldots \bowtie ess_{n-2} \bowtie srep \bowtie
\end{array}
$$

12.2.8.4 `NofixName`

$$
\begin{array}{rcl}
ln \,\_\, ess_1 \;\ldots\; \_\, ess_{n-1} \,\_\, er & \Longrightarrow & ln \bowtie ess_1 \ldots \bowtie ess_{n-1} \bowtie er \\
ln \,\_\, ess_1 \;\ldots\; \_\, ess_{n-1} \,\_\, sr & \Longrightarrow & ln \bowtie ess_1 \ldots \bowtie ess_{n-1} \bowtie sr \\
lp \,\_\, ess_1 \;\ldots\; \_\, ess_{n-1} \,\_\, erp & \Longrightarrow & ln \bowtie ess_1 \ldots \bowtie ess_{n-1} \bowtie erp \\
lp \,\_\, ess_1 \;\ldots\; \_\, ess_{n-1} \,\_\, srp & \Longrightarrow & ln \bowtie ess_1 \ldots \bowtie ess_{n-1} \bowtie srp
\end{array}
$$

12.2.9 Generic name

All generic names are transformed to juxtapositions of `NAME`s and generic parameter lists. This causes the generic operator definition paragraphs in which they appear to become generic horizontal definition paragraphs, and thus be amenable to further syntactic transformation.

Each resulting `NAME` should be one for which there is an operator template paragraph in scope (see 12.2.8).

12.2.9.1 `PrefixGenName`

$$
\begin{array}{rcl}
pre \; i & \Longrightarrow & pre \bowtie \; [i] \\
ln \; i_1 \; ess_1 \;\ldots\; i_{n-2} \; ess_{n-2} \; i_{n-1} \; ere \; i_n & \Longrightarrow & ln \bowtie ess_1 \ldots \bowtie ess_{n-2} \bowtie ere \bowtie \; [i_1, \ldots, i_{n-2}, i_{n-1}, i_n] \\
ln \; i_1 \; ess_1 \;\ldots\; i_{n-2} \; ess_{n-2} \; i_{n-1} \; sre \; i_n & \Longrightarrow & ln \bowtie ess_1 \ldots \bowtie ess_{n-2} \bowtie sre \bowtie \; [i_1, \ldots, i_{n-2}, i_{n-1}, i_n]
\end{array}
$$

12.2.9.2 `PostfixGenName`

$$
\begin{array}{rcl}
i \; post & \Longrightarrow & \bowtie post \; [i] \\
i_1 el \; i_2 \; ess_2 \;\ldots\; i_{n-1} \; ess_{n-1} \; i_n \; er & \Longrightarrow & \bowtie el \bowtie ess_2 \ldots \bowtie ess_{n-1} \bowtie er \; [i_1, i_2, \ldots, i_{n-1}, i_n] \\
i_1 el \; i_2 \; ess_2 \;\ldots\; i_{n-1} \; ess_{n-1} \; i_n \; sr & \Longrightarrow & \bowtie el \bowtie ess_2 \ldots \bowtie ess_{n-1} \bowtie sr \; [i_1, i_2, \ldots, i_{n-1}, i_n]
\end{array}
$$

12.2.9.3 `InfixGenName`

$$
\begin{array}{rcl}
i_1 in \; i_2 & \Longrightarrow & \bowtie in \bowtie \; [i_1, i_2] \\
i_1 el \; i_2 \; ess_2 \;\ldots\; i_{n-2} \; ess_{n-2} \; i_{n-1} \; ere \; i_n & \Longrightarrow & \bowtie el \bowtie ess_2 \ldots \bowtie ess_{n-2} \bowtie ere \bowtie \; [i_1, i_2, \ldots, i_{n-2}, i_{n-1}, i_n] \\
i_1 el \; i_2 \; ess_2 \;\ldots\; i_{n-2} \; ess_{n-2} \; i_{n-1} \; sre \; i_n & \Longrightarrow & \bowtie el \bowtie ess_2 \ldots \bowtie ess_{n-2} \bowtie sre \bowtie \; [i_1, i_2, \ldots, i_{n-2}, i_{n-1}, i_n]
\end{array}
$$

12.2.9.4 `NofixGenName`

$$
\begin{array}{rcl}
ln\ i_1\ ess_1\ ...\ i_{n-1}\ ess_{n-1}\ i_n\ er & \Longrightarrow & ln{\bowtie}ess_1...{\bowtie}ess_{n-1}{\bowtie}er\ [i_1,...,i_{n-1},i_n] \\
ln\ i_1\ ess_1\ ...\ i_{n-1}\ ess_{n-1}\ i_n\ sr & \Longrightarrow & ln{\bowtie}ess_1...{\bowtie}ess_{n-1}{\bowtie}sr\ [i_1,...,i_{n-1},i_n]
\end{array}
$$

12.2.10 Relation operator application

All relation operator applications are transformed to annotated membership predicates.

Each relation `NAME` should be one for which there is an operator template paragraph in scope (see 12.2.8).

The left-hand sides of many of these transformation rules involve `ExpSep` phrases: they use es metavariables. None of them use ss metavariables: in other words, only the `Expression ES` case of `ExpSep` is specified, not the `ExpressionList SS` case. Where the latter case occurs in a specification, the `ExpressionList` shall be transformed by rule 12.2.12 to an expression, and thence a transformation analogous to that specified for the former case can be performed, differing only in that a ss appears in the relation name in place of an es.

12.2.10.1 `PrefixRel`

$$
\begin{array}{rcl}
prep\ e & \Longrightarrow & e \in prep{\bowtie} \\
lp\ e_1\ es_1\ ...\ e_{n-2}\ es_{n-2}\ e_{n-1}\ erep\ e_n & \Longrightarrow & (e_1,...,e_{n-2},e_{n-1},e_n) \in lp{\bowtie}es_1...{\bowtie}es_{n-2}{\bowtie}erep{\bowtie} \\
lp\ e_1\ es_1\ ...\ e_{n-2}\ es_{n-2}\ al_{n-1}\ srep\ e_n & \Longrightarrow & (e_1,...,e_{n-2},al_{n-1},e_n) \in lp{\bowtie}es_1...{\bowtie}es_{n-2}{\bowtie}srep{\bowtie}
\end{array}
$$

12.2.10.2 `PostfixRel`

$$
\begin{array}{rcl}
e\ postp & \Longrightarrow & e \in {\bowtie}postp \\
e_1\ elp\ e_2\ es_2\ ...\ e_{n-1}\ es_{n-1}\ e_n\ erp & \Longrightarrow & (e_1,e_2,...,e_{n-1},e_n) \in {\bowtie}elp{\bowtie}es_2...{\bowtie}es_{n-1}{\bowtie}erp \\
e_1\ elp\ e_2\ es_2\ ...\ e_{n-1}\ es_{n-1}\ al_n\ srp & \Longrightarrow & (e_1,e_2,...,e_{n-1},al_n) \in {\bowtie}elp{\bowtie}es_2...{\bowtie}es_{n-1}{\bowtie}srp
\end{array}
$$

12.2.10.3 `InfixRel`

$$
e_1\ ip_1\ e_2\ ip_2\ e_3\ ... \quad \Longrightarrow \quad e_1\ ip_1\ e_2 \mathrel{⸪} \tau_1 \land e_2 \mathrel{⸪} \tau_1\ ip_2\ e_3 \mathrel{⸪} \tau_2\ ...
$$

The chained relation $e_1\ ip_1\ e_2\ ip_2\ e_3\ ...$ is semantically equivalent to a conjunction of relational predicates, with the constraint that duplicated expressions be of the same type.

$$
\begin{array}{rcl}
e_1 = e_2 & \Longrightarrow & e_1 \in \{e_2\} \\
e_1\ ip\ e_2 & \Longrightarrow & (e_1,e_2) \in {\bowtie}ip{\bowtie}
\end{array}
$$

ip in the above transformation is excluded from being $\in$ or $=$, whereas $ip_1, ip_2, ...$ can be $\in$ or $=$.

$$
\begin{array}{rcl}
e_1\ elp\ e_2\ es_2\ ...\ e_{n-2}\ es_{n-2}\ e_{n-1}\ erep\ e_n & \Longrightarrow & (e_1,e_2,...,e_{n-2},e_{n-1},e_n) \in {\bowtie}elp{\bowtie}es_2...{\bowtie}es_{n-2}{\bowtie}erep{\bowtie} \\
e_1\ elp\ e_2\ es_2\ ...\ e_{n-2}\ es_{n-2}\ al_{n-1}\ srep\ e_n & \Longrightarrow & (e_1,e_2,...,e_{n-2},al_{n-1},e_n) \in {\bowtie}elp{\bowtie}es_2...{\bowtie}es_{n-2}{\bowtie}srep{\bowtie}
\end{array}
$$

12.2.10.4 `NofixRel`

$$
\begin{array}{rcl}
lp\ e_1\ es_1\ ...\ e_{n-1}\ es_{n-1}\ e_n\ erp & \Longrightarrow & (e_1, ..., e_{n-1}, e_n) \in lp\bowtie es_1 ...\bowtie es_{n-1}\bowtie erp \\
lp\ e_1\ es_1\ ...\ e_{n-1}\ es_{n-1}\ al_n\ srp & \Longrightarrow & (e_1, ..., e_{n-1}, al_n) \in lp\bowtie es_1 ...\bowtie es_{n-1}\bowtie srp
\end{array}
$$

12.2.11 Function and generic operator application

All function operator applications are transformed to annotated application expressions.

All generic operator applications are transformed to annotated generic instantiation expressions.

Each resulting `NAME` should be one for which there is an operator template paragraph in scope (see 12.2.8).

The left-hand sides of many of these transformation rules involve `ExpSep` phrases: they use es metavariables. None of them use ss metavariables: in other words, only the `Expression ES` case of `ExpSep` is specified, not the `ExpressionList SS` case. Where the latter case occurs in a specification, the `ExpressionList` shall be transformed by rule 12.2.12 to an expression, and thence a transformation analogous to that specified for the former case can be performed, differing only in that a ss appears in the function or generic name in place of an es.

12.2.11.1 `PrefixApp`

$$
\begin{array}{rcl}
pre\ e & \Longrightarrow & pre\bowtie\ e \\
ln\ e_1\ es_1\ ...\ e_{n-2}\ es_{n-2}\ e_{n-1}\ ere\ e_n & \Longrightarrow & ln\bowtie es_1 ...\bowtie es_{n-2}\bowtie ere\bowtie\ (e_1, ..., e_{n-2}, e_{n-1}, e_n) \\
ln\ e_1\ es_1\ ...\ e_{n-2}\ es_{n-2}\ al_{n-1}\ sre\ e_n & \Longrightarrow & ln\bowtie es_1 ...\bowtie es_{n-2}\bowtie sre\bowtie\ (e_1, ..., e_{n-2}, al_{n-1}, e_n)
\end{array}
$$

$$
\begin{array}{rcl}
pre\ e & \Longrightarrow & pre\bowtie\ [e] \\
ln\ e_1\ es_1\ ...\ e_{n-2}\ es_{n-2}\ e_{n-1}\ ere\ e_n & \Longrightarrow & ln\bowtie es_1 ...\bowtie es_{n-2}\bowtie ere\bowtie\ [e_1, ..., e_{n-2}, e_{n-1}, e_n] \\
ln\ e_1\ es_1\ ...\ e_{n-2}\ es_{n-2}\ al_{n-1}\ sre\ e_n & \Longrightarrow & ln\bowtie es_1 ...\bowtie es_{n-2}\bowtie sre\bowtie\ [e_1, ..., e_{n-2}, al_{n-1}, e_n]
\end{array}
$$

12.2.11.2 `PostfixApp`

$$
\begin{array}{rcl}
e\ post & \Longrightarrow & \bowtie post\ e \\
e_1\ el\ e_2\ es_2\ ...\ e_{n-1}\ es_{n-1}\ e_n\ er & \Longrightarrow & \bowtie el\bowtie es_2 ...\bowtie es_{n-1}\bowtie er\ (e_1, e_2, ..., e_{n-1}, e_n) \\
e_1\ el\ e_2\ es_2\ ...\ e_{n-1}\ es_{n-1}\ al_n\ sr & \Longrightarrow & \bowtie el\bowtie es_2 ...\bowtie es_{n-1}\bowtie sr\ (e_1, e_2, ..., e_{n-1}, al_n)
\end{array}
$$

$$
\begin{array}{rcl}
e\ post & \Longrightarrow & \bowtie post\ [e] \\
e_1\ el\ e_2\ es_2\ ...\ e_{n-1}\ es_{n-1}\ e_n\ er & \Longrightarrow & \bowtie el\bowtie es_2 ...\bowtie es_{n-1}\bowtie er\ [e_1, e_2, ..., e_{n-1}, e_n] \\
e_1\ el\ e_2\ es_2\ ...\ e_{n-1}\ es_{n-1}\ al_n\ sr & \Longrightarrow & \bowtie el\bowtie es_2 ...\bowtie es_{n-1}\bowtie sr\ [e_1, e_2, ..., e_{n-1}, al_n]
\end{array}
$$

12.2.11.3 InfixApp

$$\begin{array}{rcl} e_1\ in\ e_2 & \Longrightarrow & \bowtie in \bowtie\ (e_1, e_2) \\ e_1\ el\ e_2\ es_2\ ...\ e_{n-2}\ es_{n-2}\ e_{n-1}\ ere\ e_n & \Longrightarrow & \bowtie el \bowtie es_2 ... \bowtie es_{n-2} \bowtie ere \bowtie\ (e_1, e_2, ..., e_{n-2}, e_{n-1}, e_n) \\ e_1\ el\ e_2\ es_2\ ...\ e_{n-2}\ es_{n-2}\ al_{n-1}\ sre\ e_n & \Longrightarrow & \bowtie el \bowtie es_2 ... \bowtie es_{n-2} \bowtie sre \bowtie\ (e_1, e_2, ..., e_{n-2}, al_{n-1}, e_n) \end{array}$$

$$\begin{array}{rcl} e_1\ in\ e_2 & \Longrightarrow & \bowtie in \bowtie\ [e_1, e_2] \\ e_1\ el\ e_2\ es_2\ ...\ e_{n-2}\ es_{n-2}\ e_{n-1}\ ere\ e_n & \Longrightarrow & \bowtie el \bowtie es_2 ... \bowtie es_{n-2} \bowtie ere \bowtie\ [e_1, e_2, ..., e_{n-2}, e_{n-1}, e_n] \\ e_1\ el\ e_2\ es_2\ ...\ e_{n-2}\ es_{n-2}\ al_{n-1}\ sre\ e_n & \Longrightarrow & \bowtie el \bowtie es_2 ... \bowtie es_{n-2} \bowtie sre \bowtie\ [e_1, e_2, ..., e_{n-2}, al_{n-1}, e_n] \end{array}$$

12.2.11.4 NofixApp

$$\begin{array}{rcl} ln\ e_1\ es_1\ ...\ e_{n-1}\ es_{n-1}\ e_n\ er & \Longrightarrow & ln \bowtie es_1 ... \bowtie es_{n-1} \bowtie er\ \ (e_1, ..., e_{n-1}, e_n) \\ ln\ e_1\ es_1\ ...\ e_{n-1}\ es_{n-1}\ al_n\ sr & \Longrightarrow & ln \bowtie es_1 ... \bowtie es_{n-1} \bowtie sr\ \ (e_1, ..., e_{n-1}, al_n) \end{array}$$

$$\begin{array}{rcl} ln\ e_1\ es_1\ ...\ e_{n-1}\ es_{n-1}\ e_n\ er & \Longrightarrow & ln \bowtie es_1 ... \bowtie es_{n-1} \bowtie er\ \ [e_1, ..., e_{n-1}, e_n] \\ ln\ e_1\ es_1\ ...\ e_{n-1}\ es_{n-1}\ al_n\ sr & \Longrightarrow & ln \bowtie es_1 ... \bowtie es_{n-1} \bowtie sr\ \ [e_1, ..., e_{n-1}, al_n] \end{array}$$

12.2.12 Expression list

$$e_1, ..., e_n \quad \Longrightarrow \quad \{(1, e_1), ..., (n, e_n)\}$$

Within an operator application, each expression list is syntactically transformed to the equivalent explicit representation of a sequence, which is a set of pairs of position and corresponding component expression.

13 Type inference rules

13.1 Introduction

All expressions in Z are typed, allowing some of the logical anomalies that can arise when sets are defined in terms of their properties to be avoided. An example of a Z phrase that is not well-typed is the predicate $2 \in 3$, because the second expression of a membership predicate is required to be a set of values, each of the same type as the first expression. The check for well-typedness of a Z specification can be automated, by conforming to the specification given in this clause.

The type constraints that shall be satisfied between the various parts of a Z phrase are specified by type inference rules, of which there is one corresponding to each annotated syntax production. The type inference rules together can be viewed as a partial function that maps a parse tree of an annotated syntax sentence to a fully annotated parse tree of an annotated syntax sentence.

Initially, all annotations are set to variables, and these are all distinct except as set by the syntactic transformation rules (12.2.7 and 12.2.10). A type inference rule's type sequent is a pattern that when matched against a phrase produces substitutions for its metavariables. Starting with a type sequent for a whole Z specification, that is matched against the pattern in the type inference rule for a sectioned specification. The resulting substitutions for metavariables are used to produce instantiations of the rule's type subsequents and side-conditions. The instantiated side-conditions include constraints that determine the environments to be used in typechecking the type subsequents. There is no need to solve the constraints yet. Instead, type inference rules can be applied to the generated type subsequents, each application producing zero or more new type subsequents, until no more type subsequents remain. This produces a tree of deductions, whose leaves correspond to the atomic phrases of the sentence, namely given types paragraphs, truth predicates, and reference expressions.

There remains a collection of constraints to be solved. There are dependencies between constraints: for example, a constraint that checks that a name is declared in an environment cannot be solved until that environment has been determined by other constraints. Unification is a suitable mechanism for solving constraints. A typechecker shall not impose any additional constraints, such as on the order in which constraints are expected to be solved [15].

For a well-typed specification, there shall be no contradictions amongst the constraints, and the solution to the constraints shall provide values for all of the variables. If there is a contradiction amongst the constraints, there can be no consistent assignment of annotations, and the specification is not well-typed. If the solution to the constraints does not provide a value for a variable, there is more than one possible assignment of annotations, and the specification is not well-typed.

EXAMPLE 1

$$\mid \ \mathit{empty} == \{\}$$

In this declaration, the type of *empty*, $\mathbb{P}\,\alpha$, involves an unconstrained variable.

13.2 Formal definition of type inference rules

13.2.1 Specification

13.2.1.1 Sectioned specification

$$\frac{\{\} \vdash^{\mathcal{S}} s_{prelude} \ {}^{\circ}_{\circ}\ \Gamma_0 \qquad \delta_1 \vdash^{\mathcal{S}} s_1 \ {}^{\circ}_{\circ}\ \Gamma_1 \qquad \ldots \qquad \delta_n \vdash^{\mathcal{S}} s_n \ {}^{\circ}_{\circ}\ \Gamma_n}{\vdash^{\mathcal{Z}} s_1 \ {}^{\circ}_{\circ}\ \Gamma_1 \ \ldots \ s_n \ {}^{\circ}_{\circ}\ \Gamma_n} \left(\begin{array}{l} \delta_1 = \{prelude \mapsto \Gamma_0\} \\ \quad\vdots \\ \delta_n = \delta_{n-1} \cup \{i_{n-1} \mapsto \Gamma_{n-1}\} \end{array} \right)$$

where i_{n-1} is the name of section s_{n-1}, and none of the sections $s_1 \ldots s_n$ are named *prelude*.

Each section is typechecked in an environment formed from preceding sections, and is annotated with an environment that it establishes.

NOTE The environment established by the prelude section is as follows.

$$\begin{array}{rcl} \Gamma_0 & = & (\mathbb{A}, (prelude, \mathbb{P}(\mathtt{GIVEN}\ \mathbb{A}))), \\ & & (\mathbb{N}, (prelude, \mathbb{P}(\mathtt{GIVEN}\ \mathbb{A}))), \\ & & (number\_literal\_0, (prelude, (\mathtt{GIVEN}\ \mathbb{A}))), \\ & & (number\_literal\_1, (prelude, (\mathtt{GIVEN}\ \mathbb{A}))), \\ & & (\bowtie{+}\bowtie, (prelude, \mathbb{P}(((\mathtt{GIVEN}\ \mathbb{A}) \times (\mathtt{GIVEN}\ \mathbb{A})) \times (\mathtt{GIVEN}\ \mathbb{A})))) \end{array}$$

If one of the sections $s_1 \ldots s_n$ is named *prelude*, then the same type inference rule applies except that the type subsequent for the prelude section is omitted.

13.2.2 Section

13.2.2.1 Inheriting section

$$\frac{\beta_0 \vdash^{\mathcal{D}} d_1 \mathbin{⨾} \sigma_1 \quad \ldots \quad \beta_{n-1} \vdash^{\mathcal{D}} d_n \mathbin{⨾} \sigma_n}{\begin{array}{c}\Lambda \vdash^{\mathcal{S}} \text{section } i \text{ parents } i_1, ..., i_m \texttt{ END} \\ d_1 \mathbin{⨾} \sigma_1 \ldots d_n \mathbin{⨾} \sigma_n \mathbin{⨾} \Gamma\end{array}} \left(\begin{array}{l} i \notin \mathit{dom}\ \Lambda \\ \{i_1, ..., i_m\} \subseteq \mathit{dom}\ \Lambda \\ \gamma_{-1} = \text{if } i = \mathit{prelude} \text{ then } \{\} \text{ else } \Lambda\ \mathit{prelude} \\ \gamma_0 = \gamma_{-1} \cup \Lambda\ i_1 \cup ... \cup \Lambda\ i_m \\ \beta_0 = \gamma_0 \mathbin{⨾} \mathit{second} \\ \mathit{disjoint}\ \langle \mathit{dom}\ \sigma_1, ..., \mathit{dom}\ \sigma_n \rangle \\ \Gamma \in (\_ \mathbin{+\!\!\!\!\!\to} \_) \\ \Gamma = \gamma_0 \cup \{j : \texttt{NAME};\ \tau : \texttt{Type} \mid j \mapsto \tau \in \sigma_1 \cup ... \cup \sigma_n \bullet j \mapsto (i, \tau)\} \\ \beta_1 = \beta_0 \cup \sigma_1 \\ \quad \vdots \\ \beta_{n-1} = \beta_{n-2} \cup \sigma_{n-1} \end{array}\right)$$

Taking the side-conditions in order, this type inference rule ensures that:

a) the name of the section, i, is different from that of any previous section;

b) the names in the parents list are names of known sections;

c) the section environment of the prelude is included if the section is not itself the prelude;

d) the section environment γ_0 is formed from those of the parents;

e) the type environment β_0 is determined from the section environment γ_0;

f) there is no global redefinition between any pair of paragraphs of the section (the sets of names in their signatures are disjoint);

g) a name which is common to the environments of multiple parents shall have originated in a common ancestral section, and a name introduced by a paragraph of this section shall not also be introduced by another paragraph or parent section (all ensured by the combined environment being a function);

h) the annotation of the section is an environment formed from those of its parents extended according to the signatures of its paragraphs;

i) and the type environment in which a paragraph is typechecked is formed from that of the parent sections extended with the signatures of the preceding paragraphs of this section.

NOTE 1 Ancestors need not be immediate parents, and a section cannot be amongst its own ancestors (no cycles in the parent relation).

NOTE 2 The name of a section can be the same as the name of a variable introduced in a declaration—the two are not confused.

13.2.3 Generic instantiation

Generic declarations can appear only at the paragraph level. The types of generic declarations shall be determined before the constraints arising from the side-conditions of the type inference rules for references to generics (13.2.6.1 and 13.2.6.2) can be solved. This implies solving the constraints in per-paragraph batches. Having determined the types of references to generic declarations, instantiations that were left implicit are made explicit, ready for subsequent semantic relation.

NOTE This is why generic instantiation is defined here, immediately before the type inference rules for paragraphs.

13.2.3.1 Generic type instantiation

The constraints that cannot be solved until the type of a generic declaration is determined are those that involve the operation of generic type instantiation. The generic type instantiation meta-function relates a known generic type and a list of argument types to the type in which each reference to a generic parameter has been substituted with the corresponding argument type. Applications of the generic type instantiation meta-function are formulated here as the juxtaposition of a generic type (parenthesized) with a square-bracketed list of argument types.

$$\begin{aligned}
([i_1, ..., i_n]\ \mathtt{GIVEN}\ i)\ [\tau_1, ..., \tau_n] &= \mathtt{GIVEN}\ i \\
([i_1, ..., i_n]\ \mathtt{GENTYPE}\ i_k)\ [\tau_1, ..., \tau_n] &= \tau_k \\
([i_1, ..., i_n]\ \mathbb{P}\,\tau)\ [\tau_1, ..., \tau_n] &= \mathbb{P}(([i_1, ..., i_n]\ \tau)\ [\tau_1, ..., \tau_n]) \\
([i_1, ..., i_n]\ \tau'_1 \times ... \times \tau'_m)\ [\tau_1, ..., \tau_n] &= ([i_1, ..., i_n]\ \tau'_1)\ [\tau_1, ..., \tau_n] \times ... \times ([i_1, ..., i_n]\ \tau'_m)\ [\tau_1, ..., \tau_n] \\
([i_1, ..., i_n]\ [i'_1 : \tau'_1;\ ...;\ i'_m : \tau'_m])\ [\tau_1, ..., \tau_n]& \\
&= [i'_1 : [i_1, ..., i_n]\ \tau'_1\ [\tau_1, ..., \tau_n];\ ...;\ i'_m : [i_1, ..., i_n]\ \tau'_m\ [\tau_1, ..., \tau_n]]
\end{aligned}$$

NOTE There is no equation for variable type because that is the case of the type of the generic declaration being unknown. In a well-typed specification, all variable types within a generic type will have been unified with other types.

13.2.3.2 Carrier set

The meta-function *carrier* relates a type phrase to an expression phrase denoting the carrier set of that type. It is used for the calculation of implicit generic actuals, and also later in semantic transformation rules.

$$\begin{aligned}
\mathit{carrier}\ (\mathtt{GIVEN}\ i) &= i \ ⨾\ \mathbb{P}(\mathtt{GIVEN}\ i) \\
\mathit{carrier}\ (\mathtt{GENTYPE}\ i) &= i \ ⨾\ \mathbb{P}(\mathtt{GENTYPE}\ i) \\
\mathit{carrier}\ (\mathbb{P}\,\tau) &= \mathbb{P}(\mathit{carrier}\ \tau) \ ⨾\ \mathbb{P}\,\mathbb{P}\,\tau \\
\mathit{carrier}\ (\tau_1 \times ... \times \tau_n) &= (\mathit{carrier}\ \tau_1 \times ... \times \mathit{carrier}\ \tau_n) \ ⨾\ \mathbb{P}(\tau_1 \times ... \times \tau_n) \\
\mathit{carrier}\ ([i_n : \tau_n;\ ...;\ i_n : \tau_n]) &= [i_n : \mathit{carrier}\ \tau_n;\ ...;\ i_n : \mathit{carrier}\ \tau_n] \ ⨾\ \mathbb{P}[i_n : \tau_n;\ ...;\ i_n : \tau_n]
\end{aligned}$$

NOTE 1 The expressions are generated with type annotations, to avoid needing to apply type inference again, and so avoid the potential problem of type names being captured by local declarations.

NOTE 2 But for the GIVEN/GENTYPE distinction and the generation of type annotations, each of these equations generates an expression that has the same textual appearance as the type.

NOTE 3 There is no equation for variable type because in a well-typed specification all variable types have been unified with other types. There is no equation for generic types because they appear in only the type annotation of generic axiomatic paragraphs, and *carrier* is never applied there.

13.2.3.3 Implicit instantiation

The value of a reference expression that refers to a generic definition is an inferred instantiation of that generic definition.

$$i \ ⨾\ [i_1,\ ...,\ i_n]\tau, \tau' \quad \overset{\tau' = ([i_1,\ ...,\ i_n]\tau)\ [\alpha_1,\ ...,\ \alpha_n]}{\Longrightarrow} \quad i\ [\mathit{carrier}\ \alpha_1, ..., \mathit{carrier}\ \alpha_n] \ ⨾\ \tau'$$

It is semantically equivalent to the generic instantiation expression whose generic actuals are the carrier sets of the types inferred for the generic parameters. The type τ' is an instantiation of the generic type τ. The types inferred for the generic parameters are α_1, ..., α_n. They shall all be determinable by comparison of τ with τ' as suggested by the condition on the transformation. Cases where these types cannot be so determined, because the generic type is independent of some of the generic parameters, are not well-typed.

EXAMPLE 1 The paragraph

$a[X] == 1$

defines a with type $[X]\mathtt{GIVEN}\ \mathbb{A}$. The paragraph

$$b == a$$

typechecks, giving the annotated expression $a \mathbin{⨾} [X]\mathtt{GIVEN}\ \mathbb{A}, \mathtt{GIVEN}\ \mathbb{A}$. Comparison of the generic type with the instantiated type does not determine a type for the generic parameter X, and so this specification is not well-typed.

Cases where these types are not unique (contain unconstrained variables) are not well-typed.

EXAMPLE 2 The paragraph

$$\mathit{empty} == \varnothing$$

will contain the annotated expression $\varnothing \mathbin{⨾} [X]\,\mathbb{P}\,X, \mathbb{P}\,\alpha$, in which the type determined for the generic parameter X is unconstrained, and so this specification is not well-typed.

13.2.4 Paragraph

13.2.4.1 Given types paragraph

$$\frac{}{\Sigma \ \vdash^{\mathcal{D}} \ [i_1, ..., i_n]\ \mathtt{END} \mathbin{⨾} \sigma} \left(\begin{array}{l} \#\,\{i_1,\ ...,\ i_n\} = n \\ \sigma = i_1 : \mathbb{P}(\mathtt{GIVEN}\ i_1);\ ...;\ i_n : \mathbb{P}(\mathtt{GIVEN}\ i_n) \end{array} \right)$$

In a given types paragraph, there shall be no duplication of names. The annotation of the paragraph is a signature associating the given type names with powerset types.

13.2.4.2 Axiomatic description paragraph

$$\frac{\Sigma \ \vdash^{\mathcal{E}} \ e \mathbin{⨾} \tau}{\Sigma \ \vdash^{\mathcal{D}} \ \mathtt{AX}\ e \mathbin{⨾} \tau\ \mathtt{END} \mathbin{⨾} \sigma} (\tau = \mathbb{P}[\sigma])$$

In an axiomatic description paragraph $\mathtt{AX}\ e\ \mathtt{END}$, the expression e shall be a schema. The annotation of the paragraph is the signature of that schema.

13.2.4.3 Generic axiomatic description paragraph

$$\frac{\Sigma \oplus \{i_1 \mapsto \mathbb{P}(\mathtt{GENTYPE}\ i_1),\ ...,\ i_n \mapsto \mathbb{P}(\mathtt{GENTYPE}\ i_n)\} \ \vdash^{\mathcal{E}} \ e \mathbin{⨾} \tau}{\Sigma \ \vdash^{\mathcal{D}} \ \mathtt{GENAX}\ [i_1, ..., i_n]\ e \mathbin{⨾} \tau\ \mathtt{END} \mathbin{⨾} \sigma} \left(\begin{array}{l} \#\,\{i_1,\ ...,\ i_n\} = n \\ \tau = \mathbb{P}[\beta] \\ \sigma = \lambda\ j : \mathit{dom}\ \beta \bullet [i_1, ..., i_n]\ (\beta\ j) \end{array} \right)$$

In a generic axiomatic description paragraph $\mathtt{GENAX}\ [i_1, ..., i_n]\ e\ \mathtt{END}$, there shall be no duplication of names within the generic parameters. The expression e is typechecked, in an environment overridden by the generic parameters, and shall be a schema. The annotation of the paragraph is formed from the signature of that schema, having the same names but associated with types that are generic.

13.2.4.4 Free types paragraph

$$
\frac{\begin{array}{l}
\beta \vdash^{\mathcal{E}} e_{1\,1} ⦂ \tau_{1\,1} \quad ... \quad \beta \vdash^{\mathcal{E}} e_{1\,n_1} ⦂ \tau_{1\,n_1} \\
\vdots \\
\beta \vdash^{\mathcal{E}} e_{r\,1} ⦂ \tau_{r\,1} \quad ... \quad \beta \vdash^{\mathcal{E}} e_{r\,n_r} ⦂ \tau_{r\,n_r}
\end{array}}{\Sigma \vdash^{\mathcal{D}} \begin{array}{l}
f_1 ::= h_{1\,1} \mid ... \mid h_{1\,m_1} \mid \\
\quad g_{1\,1}\langle\!\langle e_{1\,1} ⦂ \tau_{1\,1}\rangle\!\rangle \mid \\
\quad \vdots \mid \\
\quad g_{1\,n_1}\langle\!\langle e_{1\,n_1} ⦂ \tau_{1\,n_1}\rangle\!\rangle \ \& \\
\vdots \ \& \\
f_r ::= h_{r\,1} \mid ... \mid h_{r\,m_r} \mid \\
\quad g_{r\,1}\langle\!\langle e_{r\,1} ⦂ \tau_{r\,1}\rangle\!\rangle \mid \\
\quad \vdots \mid \\
\quad g_{r\,n_r}\langle\!\langle e_{r\,n_r} ⦂ \tau_{r\,n_r}\rangle\!\rangle
\end{array} \ \mathtt{END} ⦂ \sigma}
\left(\begin{array}{l}
\#\{f_1,\ h_{1\,1},\ ...,\ h_{1\,m_1},\ g_{1\,1},\ ...,\ g_{1\,n_1}, \\
\vdots, \\
\quad f_r,\ h_{r\,1},\ ...,\ h_{r\,m_r},\ g_{r\,1},\ ...,\ g_{r\,n_r}\} \\
\ = r + m_1 + ... + m_r + n_1 + ... + n_r \\
\beta = \Sigma \oplus \{f_1 \mapsto \mathbb{P}(\mathtt{GIVEN}\ f_1),\ ...,\ f_r \mapsto \mathbb{P}(\mathtt{GIVEN}\ f_r)\} \\
\tau_{1\,1} = \mathbb{P}\,\alpha_{1\,1} \quad ... \quad \tau_{1\,n_1} = \mathbb{P}\,\alpha_{1\,n_1} \\
\quad \vdots \\
\tau_{r\,1} = \mathbb{P}\,\alpha_{r\,1} \quad ... \quad \tau_{r\,n_r} = \mathbb{P}\,\alpha_{r\,n_r} \\
\sigma \ = \ f_1 : \mathbb{P}(\mathtt{GIVEN}\ f_1); \\
\qquad h_{1\,1} : \mathtt{GIVEN}\ f_1;\ ...;\ h_{1\,m_1} : \mathtt{GIVEN}\ f_1; \\
\qquad g_{1\,1} : \mathbb{P}(\tau_{1\,1} \times \mathtt{GIVEN}\ f_1); \\
\qquad\quad \vdots; \\
\qquad g_{1\,n_1} : \mathbb{P}(\tau_{1\,n_1} \times \mathtt{GIVEN}\ f_1); \\
\qquad \vdots; \\
\quad\ f_r : \mathbb{P}(\mathtt{GIVEN}\ f_r); \\
\qquad h_{r\,1} : \mathtt{GIVEN}\ f_r;\ ...;\ h_{r\,m_r} : \mathtt{GIVEN}\ f_r; \\
\qquad g_{r\,1} : \mathbb{P}(\tau_{r\,1} \times \mathtt{GIVEN}\ f_r); \\
\qquad\quad \vdots; \\
\qquad g_{r\,n_r} : \mathbb{P}(\tau_{r\,n_r} \times \mathtt{GIVEN}\ f_r)
\end{array}\right)
$$

In a free types paragraph, the names of the free types, elements and injections shall all be different. The expressions representing the domains of the injections are typechecked in an environment overridden by the names of the free types, and shall all be sets. The annotation of the paragraph is the signature whose names are those of all the free types, the elements, and the injections, each associated with the corresponding type.

13.2.4.5 Conjecture paragraph

$$
\frac{\Sigma \vdash^{\mathcal{P}} p}{\Sigma \vdash^{\mathcal{D}} \vdash?\ p\ \mathtt{END} ⦂ \sigma}(\sigma = \ \epsilon)
$$

In a conjecture paragraph $\vdash?\ p\ \mathtt{END}$, the predicate p shall be well-typed. The annotation of the paragraph is the empty signature.

13.2.4.6 Generic conjecture paragraph

$$
\frac{\Sigma \oplus \{i_1 \mapsto \mathbb{P}(\mathtt{GENTYPE}\ i_1),\ ...,\ i_n \mapsto \mathbb{P}(\mathtt{GENTYPE}\ i_n)\} \vdash^{\mathcal{P}} p}{\Sigma \vdash^{\mathcal{D}} [i_1, ..., i_n] \vdash?\ p\ \mathtt{END} ⦂ \sigma}
\left(\begin{array}{l}
\#\{i_1,\ ...,\ i_n\} = n \\
\sigma = \ \epsilon
\end{array}\right)
$$

In a generic conjecture paragraph $[i_1, ..., i_n] \vdash?\ p\ \mathtt{END}$, there shall be no duplication of names within the generic parameters. The predicate p shall be well-typed in an environment overridden by the generic parameters. The annotation of the paragraph is the empty signature.

13.2.5 Predicate

13.2.5.1 Membership predicate

$$
\frac{\Sigma \vdash^{\mathcal{E}} e_1 ⦂ \tau_1 \qquad \Sigma \vdash^{\mathcal{E}} e_2 ⦂ \tau_2}{\Sigma \vdash^{\mathcal{P}} (e_1 ⦂ \tau_1) \in (e_2 ⦂ \tau_2)}\left(\ \tau_2 = \mathbb{P}\,\tau_1\ \right)
$$

In a membership predicate $e_1 \in e_2$, expression e_2 shall be a set, and expression e_1 shall be of the same type as the members of set e_2.

13.2.5.2 Truth predicate

$$\frac{}{\Sigma \vdash^{\mathcal{P}} \text{true}}$$

A truth predicate is always well-typed.

13.2.5.3 Negation predicate

$$\frac{\Sigma \vdash^{\mathcal{P}} p}{\Sigma \vdash^{\mathcal{P}} \neg\, p}$$

A negation predicate $\neg\, p$ is well-typed if and only if predicate p is well-typed.

13.2.5.4 Conjunction predicate

$$\frac{\Sigma \vdash^{\mathcal{P}} p_1 \qquad \Sigma \vdash^{\mathcal{P}} p_2}{\Sigma \vdash^{\mathcal{P}} p_1 \land p_2}$$

A conjunction predicate $p_1 \land p_2$ is well-typed if and only if predicates p_1 and p_2 are well-typed.

13.2.5.5 Universal quantification predicate

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathbin{⨾} \tau \qquad \Sigma \oplus \beta \vdash^{\mathcal{P}} p}{\Sigma \vdash^{\mathcal{P}} \forall\,(e \mathbin{⨾} \tau) \bullet p} (\tau = \mathbb{P}[\beta])$$

In a universal quantification predicate $\forall\; e \bullet p$, expression e shall be a schema, and predicate p shall be well-typed in the environment overridden by the signature of schema e.

13.2.5.6 Unique existential quantification predicate

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathbin{⨾} \tau \qquad \Sigma \oplus \beta \vdash^{\mathcal{P}} p}{\Sigma \vdash^{\mathcal{P}} \exists_1\,(e \mathbin{⨾} \tau) \bullet p} (\tau = \mathbb{P}[\beta])$$

In a unique existential quantification predicate $\exists_1\; e \bullet p$, expression e shall be a schema, and predicate p shall be well-typed in the environment overridden by the signature of schema e.

13.2.6 Expression

13.2.6.1 Reference expression

In a reference expression, if the name is of the form Δi and no declaration of this name yet appears in the environment, then the following syntactic transformation is applied.

$$\Delta i \quad \overset{\Delta i \notin dom\ \Sigma}{\Longrightarrow} \quad [i;\; i\,']$$

This syntactic transformation makes the otherwise undefined name be equivalent to the corresponding schema construction expression, which is then typechecked.

Similarly, if the name is of the form Ξi and no declaration of this name yet appears in the environment, then the following syntactic transformation is applied.

$$\Xi i \quad \overset{\Xi i \notin dom\ \Sigma}{\Longrightarrow} \quad [i;\; i\,' \mid \theta\, i = \theta\, i\,']$$

NOTE 1 The Ξ notation is deliberately not defined in terms of the Δ notation.

NOTE 2 Type inference could be done without these syntactic transformations, but they are necessary steps in defining the formal semantics.

NOTE 3 Only occurrences of Δ and Ξ that are in such reference expressions are so transformed, not others such as those in the names of declarations.

$$\frac{}{\Sigma \vdash^{\mathcal{E}} i \,⨾\, \tau} \left(\begin{array}{l} i \in dom\ \Sigma \\ \tau = \text{if } \Sigma\ i = [\imath_1, ..., \imath_n]\ \alpha \text{ then } \Sigma\ i, (\Sigma\ i)\ [\alpha_1, ..., \alpha_n] \text{ else } \Sigma\ i \end{array} \right)$$

In any other reference expression i, the name i shall be associated with a type in the environment. If that type is generic, the annotation of the whole expression is a pair of both the uninstantiated type (for the Instantiation clause to determine that this is a reference to a generic definition) and the type instantiated with new distinct variable types (which the context should constrain to non-generic types). Otherwise (if the type in the environment is non-generic), that is the type of the whole expression.

NOTE 4 If the type is generic, the reference expression will be transformed to a generic instantiation expression by the rule in 13.2.3.3. That shall be done only when the implicit instantiations have been determined via constraints on the new variable types $\alpha_1, ..., \alpha_n$.

13.2.6.2 Generic instantiation expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \,⨾\, \tau_1 \quad ... \quad \Sigma \vdash^{\mathcal{E}} e_n \,⨾\, \tau_n}{\Sigma \vdash^{\mathcal{E}} i[(e_1 \,⨾\, \tau_1), ..., (e_n \,⨾\, \tau_n)] \,⨾\, \tau} \left(\begin{array}{l} i \in dom\ \Sigma \\ \Sigma\ i = [\imath_1, ..., \imath_n]\ \alpha \\ \tau_1 = \mathbb{P}\,\alpha_1 \\ \quad \vdots \\ \tau_n = \mathbb{P}\,\alpha_n \\ \tau = (\Sigma\ i)\ [\alpha_1, ..., \alpha_n] \end{array} \right)$$

In a generic instantiation expression $i\ [e_1, ..., e_n]$, the name i shall be associated with a generic type in the environment, and the expressions $e_1, ..., e_n$ shall be sets. That generic type shall have the same number of parameters as there are sets. The type of the whole expression is the instantiation of that generic type by the types of those sets' components.

NOTE The operation of generic type instantiation is defined in 13.2.3.1.

13.2.6.3 Set extension expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \,⨾\, \tau_1 \quad ... \quad \Sigma \vdash^{\mathcal{E}} e_n \,⨾\, \tau_n}{\Sigma \vdash^{\mathcal{E}} \{(e_1 \,⨾\, \tau_1), ..., (e_n \,⨾\, \tau_n)\} \,⨾\, \tau} \left(\begin{array}{l} \text{if } n > 0 \text{ then} \\ \qquad (\tau_1 = \tau_n \\ \qquad\quad \vdots \\ \qquad \tau_{n-1} = \tau_n \\ \qquad \tau = \mathbb{P}\,\tau_1) \\ \text{else } \tau = \mathbb{P}\,\alpha \end{array} \right)$$

In a set extension expression, every component expression shall be of the same type. The type of the whole expression is a powerset of the components' type, or a powerset of a variable type if there are no components. In the latter case, the variable shall be constrained by the context, otherwise the specification is not well-typed.

13.2.6.4 Set comprehension expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \,⨾\, \tau_1 \qquad \Sigma \oplus \beta \vdash^{\mathcal{E}} e_2 \,⨾\, \tau_2}{\Sigma \vdash^{\mathcal{E}} \{(e_1 \,⨾\, \tau_1) \bullet (e_2 \,⨾\, \tau_2)\} \,⨾\, \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \tau_3 = \mathbb{P}\,\tau_2 \end{array} \right)$$

In a set comprehension expression $\{e_1 \bullet e_2\}$, expression e_1 shall be a schema. The type of the whole expression is a powerset of the type of expression e_2, as determined in an environment overridden by the signature of schema e_1.

13.2.6.5 Powerset expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e \,⨾\, \tau_1}{\Sigma \vdash^{\mathcal{E}} \mathbb{P}(e \,⨾\, \tau_1) \,⨾\, \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}\,\alpha \\ \tau_2 = \mathbb{P}\,\tau_1 \end{array} \right)$$

In a powerset expression $\mathbb{P}\,e$, expression e shall be a set. The type of the whole expression is then a powerset of the type of expression e.

13.2.6.6 Tuple extension expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \,⦂\, \tau_1 \quad \ldots \quad \Sigma \vdash^{\mathcal{E}} e_n \,⦂\, \tau_n}{\Sigma \vdash^{\mathcal{E}} ((e_1 \,⦂\, \tau_1), \ldots, (e_n \,⦂\, \tau_n)) \,⦂\, \tau} \left(\tau = \tau_1 \times \ldots \times \tau_n \right)$$

In a tuple extension expression $(e_1, \ldots, e_n)$, the type of the whole expression is the Cartesian product of the types of the individual component expressions.

13.2.6.7 Tuple selection expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e \,⦂\, \tau_1}{\Sigma \vdash^{\mathcal{E}} (e \,⦂\, \tau_1) \,.\, b \,⦂\, \tau_2} ((b, \tau_2) \in \tau_1)$$

In a tuple selection expression $e \,.\, b$, the type of expression e shall be a Cartesian product, and number b shall select one of its components. The type of the whole expression is the type of the selected component.

13.2.6.8 Binding extension expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \,⦂\, \tau_1 \quad \ldots \quad \Sigma \vdash^{\mathcal{E}} e_n \,⦂\, \tau_n}{\Sigma \vdash^{\mathcal{E}} \langle\!| \, i_1 == (e_1 \,⦂\, \tau_1), \ldots, i_n == (e_n \,⦂\, \tau_n) \, |\!\rangle \,⦂\, \tau} \left(\begin{array}{l} \# \{i_1, \ldots, i_n\} = n \\ \tau = [i_1 : \tau_1; \ldots; i_n : \tau_n] \end{array} \right)$$

In a binding extension expression $\langle\!| \, i_1 == e_1, \ldots, i_n == e_n \, |\!\rangle$, there shall be no duplication amongst the bound names. The type of the whole expression is that of a binding whose signature associates the names with the types of the corresponding expressions.

13.2.6.9 Binding construction expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e \,⦂\, \tau_1}{\Sigma \vdash^{\mathcal{E}} \theta\,(e \,⦂\, \tau_1)\,{}^{*} \,⦂\, \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \forall\, i : \mathtt{NAME} \mid (i, \alpha_1) \in \beta \bullet (i \ decor\ {}^{*}, \alpha_1) \in \Sigma \land \neg\, \alpha_1 = [\imath_1, \ldots, \imath_n]\ \alpha_2 \\ \tau_2 = [\beta] \end{array} \right)$$

In a binding construction expression $\theta\, e\ {}^{*}$, the expression e shall be a schema. Every name and type pair in its signature, with the optional decoration added, should be present in the environment with a non-generic type. The type of the whole expression is that of a binding whose signature is that of the schema.

NOTE If the type in the environment were generic, semantic transformation 14.2.5.2 would produce a reference expression whose implicit instantiation is not determined by this International Standard.

13.2.6.10 Binding selection expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e \,⦂\, \tau_1}{\Sigma \vdash^{\mathcal{E}} (e \,⦂\, \tau_1) \,.\, i \,⦂\, \tau_2} \left(\begin{array}{l} \tau_1 = [\beta] \\ (i, \tau_2) \in \beta \end{array} \right)$$

In a binding selection expression $e \,.\, i$, expression e shall be a binding, and name i shall select one of its components. The type of the whole expression is the type of the selected component.

13.2.6.11 Application expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \,⦂\, \tau_1 \quad \Sigma \vdash^{\mathcal{E}} e_2 \,⦂\, \tau_2}{\Sigma \vdash^{\mathcal{E}} (e_1 \,⦂\, \tau_1)\ (e_2 \,⦂\, \tau_2) \,⦂\, \tau_3} \left(\tau_1 = \mathbb{P}(\tau_2 \times \tau_3) \right)$$

In an application expression $e_1\ e_2$, the expression e_1 shall be a set of pairs, and expression e_2 shall be of the same type as the first components of those pairs. The type of the whole expression is the type of the second components of those pairs.

13.2.6.12 Definite description expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \,⦂\, \tau_1 \quad \Sigma \oplus \beta \vdash^{\mathcal{E}} e_2 \,⦂\, \tau_2}{\Sigma \vdash^{\mathcal{E}} \mu\,(e_1 \,⦂\, \tau_1) \bullet (e_2 \,⦂\, \tau_2) \,⦂\, \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \tau_3 = \tau_2 \end{array} \right)$$

In a definite description expression $\mu\ e_1 \bullet e_2$, expression e_1 shall be a schema. The type of the whole expression is the type of expression e_2, as determined in an environment overridden by the signature of schema e_1.

13.2.6.13 Variable construction expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathrel{⨾} \tau_1}{\Sigma \vdash^{\mathcal{E}} [i : (e \mathrel{⨾} \tau_1)] \mathrel{⨾} \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}\,\alpha \\ \tau_2 = \mathbb{P}[i : \alpha] \end{array} \right)$$

In a variable construction expression $[i : e]$, expression e shall be a set. The type of the whole expression is that of a schema whose signature associates the name i with the type of a member of the set e.

13.2.6.14 Schema construction expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathrel{⨾} \tau_1 \qquad \Sigma \oplus \beta \vdash^{\mathcal{P}} p}{\Sigma \vdash^{\mathcal{E}} [(e \mathrel{⨾} \tau_1) \mid p] \mathrel{⨾} \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \tau_2 = \tau_1 \end{array} \right)$$

In a schema construction expression $[e \mid p]$, expression e shall be a schema, and predicate p shall be well-typed in an environment overridden by the signature of schema e. The type of the whole expression is the same as the type of expression e.

13.2.6.15 Schema negation expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathrel{⨾} \tau_1}{\Sigma \vdash^{\mathcal{E}} \neg\,(e \mathrel{⨾} \tau_1) \mathrel{⨾} \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \tau_2 = \tau_1 \end{array} \right)$$

In a schema negation expression $\neg\, e$, expression e shall be a schema. The type of the whole expression is the same as the type of expression e.

13.2.6.16 Schema conjunction expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \mathrel{⨾} \tau_1 \qquad \Sigma \vdash^{\mathcal{E}} e_2 \mathrel{⨾} \tau_2}{\Sigma \vdash^{\mathcal{E}} (e_1 \mathrel{⨾} \tau_1) \land (e_2 \mathrel{⨾} \tau_2) \mathrel{⨾} \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta_1] \\ \tau_2 = \mathbb{P}[\beta_2] \\ \beta_1 \approx \beta_2 \\ \tau_3 = \mathbb{P}[\beta_1 \cup \beta_2] \end{array} \right)$$

In a schema conjunction expression $e_1 \land e_2$, expressions e_1 and e_2 shall be schemas, and their signatures shall be compatible. The type of the whole expression is that of the schema whose signature is the union of those of expressions e_1 and e_2.

13.2.6.17 Schema hiding expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathrel{⨾} \tau_1}{\Sigma \vdash^{\mathcal{E}} (e \mathrel{⨾} \tau_1) \setminus (i_1, ..., i_n) \mathrel{⨾} \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \{i_1,\ ...,\ i_n\} \subseteq dom\ \beta \\ \tau_2 = \mathbb{P}[\{i_1,\ ...,\ i_n\} \mathbin{⩤} \beta] \end{array} \right)$$

In a schema hiding expression $e \setminus (i_1, ..., i_n)$, expression e shall be a schema, and the names to be hidden shall all be in the signature of that schema. The type of the whole expression is that of a schema whose signature is computed by subtracting from the signature of expression e those pairs whose names are hidden.

13.2.6.18 Schema universal quantification expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \mathrel{⨾} \tau_1 \qquad \Sigma \oplus \beta_1 \vdash^{\mathcal{E}} e_2 \mathrel{⨾} \tau_2}{\Sigma \vdash^{\mathcal{E}} \forall\,(e_1 \mathrel{⨾} \tau_1) \bullet (e_2 \mathrel{⨾} \tau_2) \mathrel{⨾} \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta_1] \\ \tau_2 = \mathbb{P}[\beta_2] \\ \beta_1 \approx \beta_2 \\ \tau_3 = \mathbb{P}[dom\ \beta_1 \mathbin{⩤} \beta_2] \end{array} \right)$$

In a schema universal quantification expression $\forall\ e_1 \bullet e_2$, expression e_1 shall be a schema, and expression e_2, in an environment overridden by the signature of schema e_1, shall also be a schema, and the signatures of these two schemas shall be compatible. The type of the whole expression is that of a schema whose signature is computed by subtracting from the signature of e_2 those pairs whose names are in the signature of e_1.

13.2.6.19 Schema unique existential quantification expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \degree\!\!\degree\ \tau_1 \qquad \Sigma \oplus \beta_1 \vdash^{\mathcal{E}} e_2 \degree\!\!\degree\ \tau_2}{\Sigma \vdash^{\mathcal{E}} \exists_1\ (e_1 \degree\!\!\degree\ \tau_1) \bullet (e_2 \degree\!\!\degree\ \tau_2) \degree\!\!\degree\ \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta_1] \\ \tau_2 = \mathbb{P}[\beta_2] \\ \beta_1 \approx \beta_2 \\ \tau_3 = \mathbb{P}[dom\ \beta_1 \lhd \beta_2] \end{array} \right)$$

In a schema unique existential quantification expression $\exists_1\ e_1 \bullet e_2$, expression e_1 shall be a schema, and expression e_2, in an environment overridden by the signature of schema e_1, shall also be a schema, and the signatures of these two schemas shall be compatible. The type of the whole expression is that of a schema whose signature is computed by subtracting from the signature of e_2 those pairs whose names are in the signature of e_1.

13.2.6.20 Schema renaming expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e \degree\!\!\degree\ \tau_1}{\Sigma \vdash^{\mathcal{E}} (e \degree\!\!\degree\ \tau_1)[j_1 \mathbin{/} i_1, ..., j_n \mathbin{/} i_n] \degree\!\!\degree\ \tau_2} \left(\begin{array}{l} \#\ \{i_1,\ ...,\ i_n\} = n \\ \tau_1 = \mathbb{P}[\beta_1] \\ \beta_2 = \{j_1 \mapsto i_1,\ ...,\ j_n \mapsto i_n\} \degree\!\!\degree\ \beta_1 \cup \{i_1,\ ...,\ i_n\} \lhd \beta_1 \\ \tau_2 = \mathbb{P}[\beta_2] \\ \beta_2 \in (\_ \twoheadrightarrow\!\!\!\!\!\!\!\!\;\;\; \_) \end{array} \right)$$

In a schema renaming expression $e\ [j_1\ /\ i_1, ..., j_n\ /\ i_n]$, there shall be no duplicates amongst the old names $i_1, ..., i_n$. Expression e shall be a schema. The type of the whole expression is that of a schema whose signature is like that of expression e but with the new names in place of corresponding old names. Declarations that are merged by the renaming shall have the same type.

NOTE Old names need not be in the signature of the schema. This is so as to permit renaming to distribute over other notations such as disjunction.

13.2.6.21 Schema precondition expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e \degree\!\!\degree\ \tau_1}{\Sigma \vdash^{\mathcal{E}} \mathrm{pre}\ (e \degree\!\!\degree\ \tau_1) \degree\!\!\degree\ \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \tau_2 = \mathbb{P}[\{i,\ j : \mathtt{NAME} \mid j \in dom\ \beta \land (j = i\ decor\ \ ' \lor j = i\ decor\ !) \bullet j\} \lhd \beta] \end{array} \right)$$

In a schema precondition expression pre e, expression e shall be a schema. The type of the whole expression is that of a schema whose signature is computed by subtracting from the signature of e those pairs whose names have primed or shrieked decorations.

13.2.6.22 Schema composition expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \degree\!\!\degree\ \tau_1 \qquad \Sigma \vdash^{\mathcal{E}} e_2 \degree\!\!\degree\ \tau_2}{\Sigma \vdash^{\mathcal{E}} (e_1 \degree\!\!\degree\ \tau_1) \fatsemi (e_2 \degree\!\!\degree\ \tau_2) \degree\!\!\degree\ \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta_1] \\ \tau_2 = \mathbb{P}[\beta_2] \\ match = \{i : dom\ \beta_2 \mid i\ decor\ \ ' \in dom\ \beta_1 \bullet i\} \\ \beta_3 = \{i : match \bullet i\ decor\ \ '\} \lhd \beta_1 \\ \beta_4 = match \lhd \beta_2 \\ \beta_3 \approx \beta_4 \\ \{i : match \bullet i \mapsto \beta_1(i\ decor\ \ ')\} \approx \{i : match \bullet i \mapsto \beta_2\ i\} \\ \tau_3 = \mathbb{P}[\beta_3 \cup \beta_4] \end{array} \right)$$

In a schema composition expression $e_1 \fatsemi e_2$, expressions e_1 and e_2 shall be schemas. Let *match* be the set of names in schema e_2 for which there are matching primed names in schema e_1. Let β_3 be the signature formed from the components of e_1 excluding the matched primed components. Let β_4 be the signature formed from the components of e_2 excluding the matched unprimed components. Signatures β_3 and β_4 shall be compatible. The types of the excluded matched pairs of components shall be the same. The type of the whole expression is that of a schema whose signature is the union of β_3 and β_4.

13.2.6.23 Schema piping expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \mathbin{⨾} \tau_1 \qquad \Sigma \vdash^{\mathcal{E}} e_2 \mathbin{⨾} \tau_2}{\Sigma \vdash^{\mathcal{E}} (e_1 \mathbin{⨾} \tau_1) \gg (e_2 \mathbin{⨾} \tau_2) \mathbin{⨾} \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta_1] \\ \tau_2 = \mathbb{P}[\beta_2] \\ match = \{i : \mathtt{NAME} \mid i\ decor\ ! \in dom\ \beta_1 \land i\ decor\ ? \in dom\ \beta_2 \bullet i\} \\ \beta_3 = \{i : match \bullet i\ decor\ !\} \lhd \beta_1 \\ \beta_4 = \{i : match \bullet i\ decor\ ?\} \lhd \beta_2 \\ \beta_3 \approx \beta_4 \\ \{i : match \bullet i \mapsto \beta_1(i\ decor\ !)\} \approx \{i : match \bullet i \mapsto \beta_2\ (i\ decor\ ?)\} \\ \tau_3 = \mathbb{P}[\beta_3 \cup \beta_4] \end{array} \right)$$

In a schema piping expression $e_1 \gg e_2$, expressions e_1 and e_2 shall be schemas. Let *match* be the set of names for which there are matching shrieked names in schema e_1 and queried names in schema e_2. Let β_3 be the signature formed from the components of e_1 excluding the matched shrieked components. Let β_4 be the signature formed from the components of e_2 excluding the matched queried components. Signatures β_3 and β_4 shall be compatible. The types of the excluded matched pairs of components shall be the same. The type of the whole expression is that of a schema whose signature is the union of β_3 and β_4.

13.2.6.24 Schema decoration expression

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathbin{⨾} \tau_1}{\Sigma \vdash^{\mathcal{E}} (e \mathbin{⨾} \tau_1)\ ^+ \mathbin{⨾} \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \tau_2 = \mathbb{P}[\{i : dom\ \beta \bullet i\ decor\ ^+ \mapsto \beta\ i\}] \end{array} \right)$$

In a schema decoration expression $e\ ^+$, expression e shall be a schema. The type of the whole expression is that of a schema whose signature is like that of e but with the stroke appended to each of its names.

13.3 Summary of scope rules

NOTE Here is an informal explanation of the scope rules implied by the type inference rules of 13.2.

A scope is static: it depends on only the structure of the text, not on the value of any predicate or expression.

A declaration can be either: a given type, a free type, a formal generic parameter, or an instance of `Declaration` usually within a `DeclPart`.

The scopes of given types and free types (which occur only at paragraph level), and `Declaration`s at paragraph level (such as those of schema definitions and those of the outermost `DeclPart` in axiomatic descriptions), are the whole of the rest of the section and any sections of which that is an ancestor.

Redeclaration at paragraph level of any name already declared at paragraph level is prohibited. Redeclaration at an inner level of any name already declared with larger scope makes a hole in the scope of the outer declaration.

In a free types paragraph, the scopes of the declarations of the free types include the right-hand sides of the free type declarations, whereas the scopes of the declarations of the elements and injections of the free types do not include the free types paragraph itself.

The scope of a formal generic parameter is the rest of the paragraph in which it appears.

A `DeclPart` is not in the scope of its declarations.

The declarations of a schema inclusion declaration are distinct from those in the signature of the schema itself, and so have separate scopes.

A name may be declared more than once within a `DeclPart` provided the types of the several declarations are identical. In this case, the declarations are merged, so that they share the same scope, and the corresponding properties are conjoined.

The scope of the declarations in the `DeclPart` of a quantification, set comprehension, function construction, definite description or schema construction expression is the | part of the `SchemaText` and any • part of that construct.

14 Semantic transformation rules

14.1 Introduction

The semantic transformation rules define some annotated notations as being equivalent to other annotated notations. The only sentences of concern here are ones that are already known to be well-formed syntactically and well-typed. These semantic transformations are transformations that could not appear earlier as syntactic transformations because they depend on type annotations or generic instantiations or are applicable only to parse trees of phrases that are not in the concrete syntax.

Some semantic transformation rules generate other transformable notation, though exhaustive application of these rules always terminates. They introduce no type errors. It is not intended that type inference be repeated on the generated notation, though type annotations are needed on that notation for the semantic relations. Nevertheless, the manipulation of type annotations is not made explicit throughout these rules, as that would be obfuscatory and can easily be derived by the reader. Indeed, some rules exploit concrete notation for brevity and clarity.

The semantic transformation rules are listed in the same order as the corresponding productions of the annotated syntax.

All applications of transformation rules that generate new declarations shall choose the names of those declarations to be such that they do not capture references.

14.2 Formal definition of semantic transformation rules

14.2.1 Specification

There are no semantic transformation rules for specifications.

14.2.2 Section

There are no semantic transformation rules for sections.

14.2.3 Paragraph

14.2.3.1 Free types paragraph

A free types paragraph is semantically equivalent to the sequence of given type paragraph and axiomatic definition paragraph defined here.

NOTE 1 This exploits notation that is not present in the annotated syntax for the purpose of abbreviation.

$f_1 ::= h_{1\,1} \mid ... \mid h_{1\,m_1} \mid g_{1\,1}\langle\!\langle e_{1\,1}\rangle\!\rangle \mid ... \mid g_{1\,n_1}\langle\!\langle e_{1\,n_1}\rangle\!\rangle$
$\&\ ...\ \&$
$f_r ::= h_{r\,1} \mid ... \mid h_{r\,m_r} \mid g_{r\,1}\langle\!\langle e_{r\,1}\rangle\!\rangle \mid ... \mid g_{r\,n_r}\langle\!\langle e_{r\,n_r}\rangle\!\rangle$

$\Longrightarrow$

$[f_1, ..., f_r]$
`END`


```
AX
```

$$
\begin{array}{l}
h_{1\,1},\ ...,\ h_{1\,m_1} : f_1 \\
\vdots \\
h_{r\,1},\ ...,\ h_{r\,m_r} : f_r \\
g_{1\,1} : \mathbb{P}(e_{1\,1} \times f_1);\ ...;\ g_{1\,n_1} : \mathbb{P}(e_{1\,n_1} \times f_1) \\
\vdots \\
g_{r\,1} : \mathbb{P}(e_{r\,1} \times f_r);\ ...;\ g_{r\,n_r} : \mathbb{P}(e_{r\,n_r} \times f_r) \\
| \\
(\forall\ u : e_{1\,1} \bullet \exists_1\ x : g_{1\,1} \bullet x\ .\ 1 = u) \land ... \land (\forall\ u : e_{1\,n_1} \bullet \exists_1\ x : g_{1\,n_1} \bullet x\ .\ 1 = u) \\
\vdots \land \\
(\forall\ u : e_{r\,1} \bullet \exists_1\ x : g_{r\,1} \bullet x\ .\ 1 = u) \land ... \land (\forall\ u : e_{r\,n_r} \bullet \exists_1\ x : g_{r\,n_r} \bullet x\ .\ 1 = u) \\
\\
(\forall\ u, v : e_{1\,1} \mid g_{1\,1} u = g_{1\,1} v \bullet u = v) \land ... \land (\forall\ u, v : e_{1\,n_1} \mid g_{1\,n_1} u = g_{1\,n_1} v \bullet u = v) \\
\vdots \land \\
(\forall\ u, v : e_{r\,1} \mid g_{r\,1} u = g_{r\,1} v \bullet u = v) \land ... \land (\forall\ u, v : e_{r\,n_r} \mid g_{r\,n_r} u = g_{r\,n_r} v \bullet u = v) \\
\\
\forall\ b_1, b_2 : \mathbb{N} \bullet \\
\quad (\forall\ w : f_1 \mid \\
\quad\quad (b_1 = 1 \land w = h_{1\,1} \lor ... \lor b_1 = m_1 \land w = h_{1\,m_1} \lor \\
\quad\quad\quad b_1 = m_1 + 1 \land w \in \{x : g_{1\,1} \bullet x\ .\ 2\} \lor ... \lor b_1 = m_1 + n_1 \land w \in \{x : g_{1\,n_1} \bullet x\ .\ 2\}) \\
\quad\quad \land (b_2 = 1 \land w = h_{1\,1} \lor ... \lor b_2 = m_1 \land w = h_{1\,m_1} \lor \\
\quad\quad\quad b_2 = m_1 + 1 \land w \in \{x : g_{1\,1} \bullet x\ .\ 2\} \lor ... \lor b_2 = m_1 + n_1 \land w \in \{x : g_{1\,n_1} \bullet x\ .\ 2\}) \bullet \\
\quad\quad\quad\quad b_1 = b_2) \land \\
\quad \vdots \land \\
\quad (\forall\ w : f_r \mid \\
\quad\quad (b_1 = 1 \land w = h_{r\,1} \lor ... \lor b_1 = m_r \land w = h_{r\,m_r} \lor \\
\quad\quad\quad b_1 = m_r + 1 \land w \in \{x : g_{r\,1} \bullet x\ .\ 2\} \lor ... \lor b_1 = m_r + n_r \land w \in \{x : g_{r\,n_r} \bullet x\ .\ 2\}) \\
\quad\quad \land (b_2 = 1 \land w = h_{r\,1} \lor ... \lor b_2 = m_r \land w = h_{r\,m_r} \lor \\
\quad\quad\quad b_2 = m_r + 1 \land w \in \{x : g_{r\,1} \bullet x\ .\ 2\} \lor ... \lor b_2 = m_r + n_r \land w \in \{x : g_{r\,n_r} \bullet x\ .\ 2\}) \bullet \\
\quad\quad\quad\quad b_1 = b_2) \\
\\
\forall\ w_1 : \mathbb{P}\, f_1;\ ...;\ w_r : \mathbb{P}\, f_r \mid \\
\quad\quad h_{1\,1} \in w_1 \land ... \land h_{1\,m_1} \in w_1 \land \\
\quad\quad \vdots \land \\
\quad\quad h_{r\,1} \in w_r \land ... \land h_{r\,m_r} \in w_r \land \\
\quad\quad (\forall\ y : (\mu\ f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{1\,1}) \bullet g_{1\,1} y \in w_1) \land \\
\quad\quad ... \land (\forall\ y : (\mu\ f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{1\,n_1}) \bullet g_{1\,n_1} y \in w_1) \land \\
\quad\quad \vdots \land \\
\quad\quad (\forall\ y : (\mu\ f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{r\,1}) \bullet g_{r\,1} y \in w_r) \land \\
\quad\quad ... \land (\forall\ y : (\mu\ f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{r\,n_r}) \bullet g_{r\,n_r} y \in w_r) \bullet \\
\quad w_1 = f_1 \land ... \land w_r = f_r
\end{array}
$$

```
END
```

The type names are introduced by the given types paragraph. The elements are declared as members of their corresponding free types. The injections are declared as functions from values in their domains to their corresponding free type.

The first of the four blank-line separated predicates is the total functionality property. It ensures that for every injection, the injection is functional at every value in its domain.

The second of the four blank-line separated predicates is the injectivity property. It ensures that for every injection, any pair of values in its domain for which the injection returns the same value shall be a pair of equal values (hence the name injection).

The third of the four blank-line separated predicates is the disjointness property. It ensures that for every free type, every pair of values of the free type are equal only if they are the same element or are returned by application of the same injection to equal values.

The fourth of the four blank-line separated predicates is the induction property. It ensures that for every free type, its members are its elements, the values returned by its injections, and nothing else.

The generated μ expressions in the induction property are intended to effect substitutions of all references to the free type names, including any such references within generic instantiation lists in the e expressions.

NOTE 2 That is why this is a semantic transformation not a syntactic one: all implicit generic instantiations shall have been made explicit before it is applied.

NOTE 3 The right-hand side of this transformation could have been expressed using the following notation from the mathematical toolkit, but for the desire to define the core language separately from the mathematical toolkit.

$$
\begin{array}{l}
[f_1, ..., f_r] \\
\texttt{END} \\
\\
\texttt{AX} \\
h_{1\,1},\ ...,\ h_{1\,m_1} : f_1 \\
\vdots \\
h_{r\,1},\ ...,\ h_{r\,m_r} : f_r \\
g_{1\,1} : e_{1\,1} \rightarrowtail f_1;\ ...;\ g_{1\,n_1} : e_{1\,n_1} \rightarrowtail f_1 \\
\vdots \\
g_{r\,1} : e_{r\,1} \rightarrowtail f_r;\ ...;\ g_{r\,n_r} : e_{r\,n_r} \rightarrowtail f_r \\
| \\
disjoint \langle \{h_{1\,1}\},\ ...,\ \{h_{1\,m_1}\},\ ran\, g_{1\,1},\ ...,\ ran\, g_{1\,n_1} \rangle \\
\vdots \\
disjoint \langle \{h_{r\,1}\},\ ...,\ \{h_{r\,m_r}\},\ ran\, g_{r\,1},\ ...,\ ran\, g_{r\,n_r} \rangle \\
\forall\ w_1 : \mathbb{P}\, f_1;\ ...;\ w_r : \mathbb{P}\, f_r\ | \\
\qquad\qquad \{h_{1\,1},\ ...,\ h_{1\,m_1}\} \cup g_{1\,1} (\!|\ \mu\, f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{1\,1}\ |\!) \\
\qquad\qquad\qquad\qquad \cup ... \cup g_{1\,n_1} (\!|\ \mu\, f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{1\,n_1}\ |\!) \subseteq w_1 \wedge \\
\qquad\qquad \vdots \wedge \\
\qquad\qquad \{h_{r\,1},\ ...,\ h_{r\,m_r}\} \cup g_{r\,1} (\!|\ \mu\, f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{r\,1}\ |\!) \\
\qquad\qquad\qquad\qquad \cup ... \cup g_{r\,n_r} (\!|\ \mu\, f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{r\,n_r}\ |\!) \subseteq w_r \bullet \\
\qquad w_1 = f_1 \wedge ... \wedge w_r = f_r \\
\texttt{END}
\end{array}
$$

14.2.4 Predicate

14.2.4.1 Unique existential predicate

The unique existential quantification predicate $\exists_1\ e \bullet p$ is *true* if and only if there is exactly one value for e for which p is *true*.

$$\exists_1\ e \bullet p \quad \Longrightarrow \quad \neg\, (\forall\ e \bullet \neg\, (p \wedge (\forall\, [e \mid p]^{\bowtie} \bullet \theta\, e = \theta\, e^{\bowtie})))$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$\exists_1\ e \bullet p \quad \Longrightarrow \quad \exists\ e \bullet p \land (\forall\ [e \mid p]^{\bowtie} \bullet \theta\, e = \theta\, e^{\ \bowtie})$$

It is semantically equivalent to there existing at least one value for e for which p is *true* and all those values for which it is *true* being the same.

14.2.5 Expression

14.2.5.1 Tuple selection expression

The value of the tuple selection expression $e\ .\ b$ is the b'th component of the tuple that is the value of e.

$$(e\ ⨾\ \tau_1 \times ... \times \tau_n)\ .\ b \quad \Longrightarrow \quad \{i : \mathit{carrier}\ (\tau_1 \times ... \times \tau_n) \bullet (i, \mu\ i_1 : \mathit{carrier}\ \tau_1;\ ...;\ i_n : \mathit{carrier}\ \tau_n \mid i = (i_1, ..., i_n) \bullet i_b)\}\ e$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$(e\ ⨾\ \tau_1 \times ... \times \tau_n)\ .\ b \quad \Longrightarrow \quad (\lambda\ i : \mathit{carrier}\ (\tau_1 \times ... \times \tau_n) \bullet \mu\ i_1 : \mathit{carrier}\ \tau_1;\ ...;\ i_n : \mathit{carrier}\ \tau_n \mid i = (i_1, ..., i_n) \bullet i_b)\ e$$

It is semantically equivalent to the function construction, from tuples of the Cartesian product type to the selected component of the tuple b, applied to the particular tuple e.

14.2.5.2 Binding construction expression

The value of the binding construction expression $\theta\, e\ ^*$ is the binding whose names are those in the signature of schema e and whose values are those of the same names with the optional decoration appended.

$$\theta\, e\ ^*\ ⨾\ [i_1 : \tau_1;\ ...;\ i_n : \tau_n] \quad \Longrightarrow \quad \langle\!|\ i_1 == i_1\ \mathit{decor}\ ^*, ..., i_n == i_n\ \mathit{decor}\ ^*\ |\!\rangle$$

It is semantically equivalent to the binding extension expression whose value is that binding.

14.2.5.3 Binding selection expression

The value of the binding selection expression $e\ .\ i$ is that value associated with i in the binding that is the value of e.

$$(e\ ⨾\ [\sigma])\ .\ i \quad \Longrightarrow \quad \{\mathit{carrier}\ [\sigma] \bullet (\mathit{chartuple}\ (\mathit{carrier}\ \ [\sigma]), i)\}\ e$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$(e\ ⨾\ [\sigma])\ .\ i \quad \Longrightarrow \quad (\lambda\ \mathit{carrier}\ [\sigma] \bullet i)\ e$$

It is semantically equivalent to the function construction expression, from bindings of the schema type of e, to the value of the selected name i, applied to the particular binding e.

14.2.5.4 Application expression

The value of the application expression $e_1\ e_2$ is the sole value associated with e_2 in the relation e_1.

$$e_1\ e_2\ ⨾\ \tau \quad \Longrightarrow \quad (\mu\ i : \mathit{carrier}\ \tau \mid (e_2, i) \in e_1 \bullet i)$$

It is semantically equivalent to that sole range value i such that the pair (e_2, i) is in the set of pairs that is the value of e_1. If there is no value or more than one value associated with e_2, then the application expression has a value but what it is is not specified.

14.2.5.5 Schema hiding expression

The value of the schema hiding expression $e \setminus (i_1, ..., i_n)$ is that schema whose signature is that of schema e minus the hidden names, and whose bindings have the same values as those in schema e.

$$(e \mathbin{⨾} \mathbb{P}[\sigma]) \setminus (i_1, ..., i_n) \quad \Longrightarrow \quad \neg\,(\forall\ i_1 : carrier\ (\sigma\ i_1);\ ...;\ i_n : carrier\ (\sigma\ i_n) \bullet \neg\ e)$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$(e \mathbin{⨾} \mathbb{P}[\sigma]) \setminus (i_1, ..., i_n) \quad \Longrightarrow \quad \exists\ i_1 : carrier\ (\sigma\ i_1);\ ...;\ i_n : carrier\ (\sigma\ i_n) \bullet e$$

It is semantically equivalent to the schema existential quantification of the hidden names $i_1, ..., i_n$ from the schema e.

14.2.5.6 Schema unique existential quantification expression

The value of the schema unique existential quantification expression $\exists_1\ e_1 \bullet e_2$ is the set of bindings of schema e_2 restricted to exclude names that are in the signature of e_1, for at least one binding of the schema e_1.

$$\exists_1\ e_1 \bullet e_2 \quad \Longrightarrow \quad \neg\,(\forall\ e_1 \bullet \neg\,(e_2 \land (\forall\,[e_1 \mid e_2]^{\bowtie} \bullet \theta\, e_1 = \theta\, e_1{}^{\bowtie})))$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$\exists_1\ e_1 \bullet e_2 \quad \Longrightarrow \quad \exists\ e_1 \bullet e_2 \land (\forall\,[e_1 \mid e_2]^{\bowtie} \bullet \theta\, e_1 = \theta\, e_1{}^{\bowtie})$$

It is semantically equivalent to a schema existential quantification expression, analogous to the unique existential quantification predicate transformation.

14.2.5.7 Schema precondition expression

The value of the schema precondition expression $\mathrm{pre}\, e$ is that schema which is like schema e but without its primed and shrieked components.

$$\mathrm{pre}(e \mathbin{⨾} \mathbb{P}[\sigma_1]) \mathbin{⨾} \mathbb{P}[\sigma_2] \quad \Longrightarrow \quad \neg\,(\forall\ carrier\ [\sigma_1 \setminus \sigma_2] \bullet \neg\ e)$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$\mathrm{pre}(e \mathbin{⨾} \mathbb{P}[\sigma_1]) \mathbin{⨾} \mathbb{P}[\sigma_2] \quad \Longrightarrow \quad \exists\ carrier\ [\sigma_1 \setminus \sigma_2] \bullet e$$

It is semantically equivalent to the existential quantification of the primed and shrieked components from the schema e.

14.2.5.8 Schema composition expression

The value of the schema composition expression $e_1 \mathbin{⨟} e_2$ is that schema representing the operation of doing the operations represented by schemas e_1 and e_2 in sequence.

$$\begin{array}{l}
(e_1 \mathbin{⨾} \mathbb{P}[\sigma_1]) \mathbin{⨟} (e_2 \mathbin{⨾} \mathbb{P}[\sigma_2]) \mathbin{⨾} \mathbb{P}[\sigma] \\
\Longrightarrow \\
\neg\,(\forall\ e^{\bowtie} \bullet \neg\,(\neg\,(\forall\ e_3 \bullet \neg\,[e_1;\ e^{\bowtie} \mid \theta\, e_3 = \theta\, e^{\bowtie}]) \\
\qquad \land \neg\,(\forall\ e_4 \bullet \neg\,[e_2;\ e^{\bowtie} \mid \theta\, e_4 = \theta\, e^{\bowtie}]))) \\
\mathit{where}\ e_3 == carrier\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i\ decor\ \ ' \mapsto \tau \in \sigma_1 \bullet i\ decor\ \ ' \mapsto \tau\}] \\
\quad \mathit{and}\ e_4 == carrier\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i \mapsto \tau \in \sigma_2 \bullet i \mapsto \tau\}] \\
\quad \mathit{and}\ e^{\bowtie} == (e_4)^{\bowtie}
\end{array}$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$
\begin{array}{l}
(e_1 \mathbin{⦂} \mathbb{P}[\sigma_1]) \mathbin{⨾} (e_2 \mathbin{⦂} \mathbb{P}[\sigma_2]) \mathbin{⦂} \mathbb{P}[\sigma] \\
\Longrightarrow \\
\exists\ e^{\bowtie} \bullet (\exists\ e_3 \bullet [e_1;\ e^{\bowtie} \mid \theta\ e_3 = \theta\ e^{\bowtie}]) \\
\quad \land (\exists\ e_4 \bullet [e_2;\ e^{\bowtie} \mid \theta\ e_4 = \theta\ e^{\bowtie}]) \\
\mathit{where}\ e_3 == \mathit{carrier}\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i\ \mathit{decor}\ \ ' \mapsto \tau \in \sigma_1 \bullet i\ \mathit{decor}\ \ ' \mapsto \tau\}] \\
\quad \mathit{and}\ e_4 == \mathit{carrier}\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i \mapsto \tau \in \sigma_2 \bullet i \mapsto \tau\}] \\
\quad \mathit{and}\ e^{\bowtie} == (e_4)^{\bowtie}
\end{array}
$$

It is semantically equivalent to the existential quantification of the matched pairs of primed components of e_1 and unprimed components of e_2, with those matched pairs being equated.

14.2.5.9 Schema piping expression

The value of the schema piping expression $e_1 \gg e_2$ is that schema representing the operation formed from the two operations represented by schemas e_1 and e_2 with the outputs of e_1 identified with the inputs of e_2.

$$
\begin{array}{l}
(e_1 \mathbin{⦂} \mathbb{P}[\sigma_1]) \gg (e_2 \mathbin{⦂} \mathbb{P}[\sigma_2]) \mathbin{⦂} \mathbb{P}[\sigma] \\
\Longrightarrow \\
\neg\ (\forall\ e^{\bowtie} \bullet \neg\ (\neg\ (\forall\ e_3 \bullet \neg\ [e_1;\ e^{\bowtie} \mid \theta\ e_3 = \theta\ e^{\bowtie}]) \\
\qquad \land \neg\ (\forall\ e_4 \bullet \neg\ [e_2;\ e^{\bowtie} \mid \theta\ e_4 = \theta\ e^{\bowtie}]))) \\
\mathit{where}\ e_3 == \mathit{carrier}\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i\ \mathit{decor}\ ! \mapsto \tau \in \sigma_1 \bullet i\ \mathit{decor}\ ! \mapsto \tau\}] \\
\quad \mathit{and}\ e_4 == \mathit{carrier}\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i\ \mathit{decor}\ ? \mapsto \tau \in \sigma_2 \bullet i\ \mathit{decor}\ ? \mapsto \tau\}] \\
\quad \mathit{and}\ e^{\bowtie} == (e_4)^{\bowtie}
\end{array}
$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$
\begin{array}{l}
(e_1 \mathbin{⦂} \mathbb{P}[\sigma_1]) \gg (e_2 \mathbin{⦂} \mathbb{P}[\sigma_2]) \mathbin{⦂} \mathbb{P}[\sigma] \\
\Longrightarrow \\
\exists\ e^{\bowtie} \bullet (\exists\ e_3 \bullet [e_1;\ e^{\bowtie} \mid \theta\ e_3 = \theta\ e^{\bowtie}]) \\
\quad \land (\exists\ e_4 \bullet [e_2;\ e^{\bowtie} \mid \theta\ e_4 = \theta\ e^{\bowtie}]) \\
\mathit{where}\ e_3 == \mathit{carrier}\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i\ \mathit{decor}\ ! \mapsto \tau \in \sigma_1 \bullet i\ \mathit{decor}\ ! \mapsto \tau\}] \\
\quad \mathit{and}\ e_4 == \mathit{carrier}\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i\ \mathit{decor}\ ? \mapsto \tau \in \sigma_2 \bullet i\ \mathit{decor}\ ? \mapsto \tau\}] \\
\quad \mathit{and}\ e^{\bowtie} == (e_4)^{\bowtie}
\end{array}
$$

It is semantically equivalent to the existential quantification of the matched pairs of shrieked components of e_1 and queried components of e_2, with those matched pairs being equated.

14.2.5.10 Schema decoration expression

The value of the schema decoration expression $e\ ^+$ is that schema whose bindings are like those of the schema e except that their names have the addition stroke $^+$.

$$(e \mathbin{⦂} \mathbb{P}[i_1 : \tau_1;\ ...;\ i_n : \tau_n])^+ \quad \Longrightarrow \quad e\ [i_1\ \mathit{decor}\ ^+ \ /\ i_1, ..., i_n\ \mathit{decor}\ ^+ \ /\ i_n]$$

It is semantically equivalent to the schema renaming where decorated names rename the original names.

15 Semantic relations

15.1 Introduction

The semantic relations define the meaning of the remaining annotated notation (that not defined by semantic transformation rules) by relation to sets of models in ZF set theory. The only sentences of concern here are ones that are already known to be well-formed syntactically and well-typed.

This clause defines the meaning of a Z specification in terms of the semantic values that its global variables may take consistent with the constraints imposed on them by the specification.

This definition is loose: it leaves the values of ill-formed definite description expressions undefined. It is otherwise tight: it specifies the values of all expressions that do not depend on values of ill-formed definite descriptions, every predicate is either *true* or *false*, and every expression denotes a value. The looseness leaves the values of undefined expressions unspecified. Any particular semantics conforms to this International Standard if it is consistent with this loose definition.

EXAMPLE The predicate $(\mu\, x : \{\ \}) \in T$ could be either *true* or *false* depending on the treatment of undefinedness.

NOTE 1 Typical specifications contain expressions that in some circumstances have undefined values. In those circumstances, those expressions ought not to affect the meaning of the specification. This definition is then sufficiently tight.

NOTE 2 Alternative treatments of undefined expressions include one or more bottoms outside of the carrier sets, or undetermined values from within the carrier sets.

15.2 Formal definition of semantic relations

15.2.1 Specification

15.2.1.1 Sectioned specification

$$[\![\ s_1\ \dots\ s_n\ ]\!]^{\mathcal{Z}} \;=\; ([\![\ \text{section}\ \ \mathit{prelude}\dots\ ]\!]^{\mathcal{S}} \fatsemi [\![\ s_1\ ]\!]^{\mathcal{S}} \fatsemi \dots \fatsemi [\![\ s_n\ ]\!]^{\mathcal{S}})\ \varnothing$$

The meaning of the Z specification $s_1\ \dots\ s_n$ is the function from sections' names to their sets of models formed by starting with the empty function and extending that with a maplet from a section's name to its set of models for each section in the specification, starting with the prelude.

To determine $[\![\ \text{section}\ \ \mathit{prelude}\dots\ ]\!]^{\mathcal{Z}}$ another prelude shall not be prefixed onto it.

NOTE The meaning of a specification is not the meaning of its last section, so as to permit several meaningful units within a single document.

15.2.2 Section

15.2.2.1 Inheriting section

The prelude section, as defined in clause 11, is treated specially, as it is the only one that does not have prelude as an implicit parent.

$$\begin{gathered}[\![\ \text{section}\ \ \mathit{prelude}\ \ \text{parents}\ \texttt{END}\ d_1\ \dots\ d_n\ ]\!]^{\mathcal{S}} \\ = \\ \lambda\ T : \mathit{SectionModels} \bullet \{\mathit{prelude} \mapsto ([\![\ d_1\ ]\!]^{\mathcal{D}} \fatsemi \dots \fatsemi [\![\ d_n\ ]\!]^{\mathcal{D}})\ (\!|\ \{\varnothing\}\ |\!)\}\end{gathered}$$

The meaning of the prelude section is given by that constant function which, whatever function from sections' names and their sets of models it is given, returns the singleton set mapping the name *prelude* to its set of models. The set of models is that to which the set containing an empty model is related by the composition of the relations between models that denote the meanings of each of the prelude's paragraphs—see clause 11 for details of those paragraphs.

NOTE One model of the prelude section can be written as follows.

$$\begin{array}{l}\{\mathbb{A} \mapsto \mathbb{A}, \\ \mathbb{N} \mapsto \mathbb{N}, \\ \mathit{number\_literal\_}0 \mapsto 0, \\ \mathit{number\_literal\_}1 \mapsto 1, \\ \_+\_ \mapsto \{((0,0),0),((0,1),1),((1,0),1),((1,1),2),\dots\}\}\end{array}$$

The behaviour of $(\_+\_)$ on non-natural numbers, e.g. reals, has not been defined at this point, so the set of models for the prelude section includes alternatives for every possible extended behaviour of addition.

$$[\![\ \text{section}\ i\ \text{ parents}\ i_1, ..., i_m\ \texttt{END}\ d_1\ ...\ d_n\ ]\!]^{\mathcal{S}}$$
$$=$$
$$\lambda\ T : \mathit{SectionModels} \bullet T \cup \{i \mapsto$$
$$([\![\ d_1\ ]\!]^{\mathcal{D}} \mathbin{⨾} ... \mathbin{⨾} [\![\ d_n\ ]\!]^{\mathcal{D}})\ (\!|\ \{M_0 : T\ \mathit{prelude};\ M_1 : T\ i_1;\ ...;\ M_m : T\ i_m;\ M : \mathit{Model} \mid M = M_0 \cup M_1 \cup ... \cup M_m \bullet M\}\ |\!)\}$$

The meaning of a section other than the prelude is the extension of a function from sections' names to their sets of models with a maplet from the given section's name to its set of models. The given section's set of models is that to which the union of the models of the section's parents is related by the composition of the relations between models that denote the meanings of each of the section's paragraphs.

15.2.3 Paragraph

15.2.3.1 Given types paragraph

The given types paragraph $[i_1, ..., i_n]$ END introduces unconstrained global names.

$$\begin{aligned}[\![\ [i_1, ..., i_n]\ \texttt{END}\ ]\!]^{\mathcal{D}} \ = \ & \{M : \mathit{Model};\ w_1,\ ...,\ w_n : \mathbb{W} \\ & \bullet M \mapsto M \cup \{i_1 \mapsto w_1,\ ...,\ i_n \mapsto w_n\} \\ & \qquad \cup \{i_1\ \mathit{decor}\ \heartsuit \mapsto w_1,\ ...,\ i_n\ \mathit{decor}\ \heartsuit \mapsto w_n\}\}\end{aligned}$$

It relates a model M to that model extended with associations between the names of the given types and semantic values chosen to represent their carrier sets. Associations for names decorated with the reserved stroke $\heartsuit$ are also introduced, so that references to them from given types (15.2.6.1) can avoid being captured.

15.2.3.2 Axiomatic description paragraph

The axiomatic description paragraph AX e END introduces global names and constraints on their values.

$$[\![\ \texttt{AX}\ e\ \texttt{END}\ ]\!]^{\mathcal{D}} \ = \ \{M : \mathit{Model};\ t : \mathbb{W} \mid t \in [\![\ e\ ]\!]^{\mathcal{E}} M \bullet M \mapsto M \cup t\}$$

It relates a model M to that model extended with a binding t of the schema that is the value of e in model M.

15.2.3.3 Generic axiomatic description paragraph

The generic axiomatic description paragraph GENAX $[i_1, ..., i_n]$ e END introduces global names and constraints on their values, with generic parameters that have to be instantiated (by sets) whenever those names are referenced.

$$\begin{aligned}&[\![\ \texttt{GENAX}\ [i_1, ..., i_n]\ (e \mathbin{⨾} \mathbb{P}[j_1 : \tau_1;\ ...;\ j_m : \tau_m])\ \texttt{END}\ ]\!]^{\mathcal{D}} = \\ &\quad \{M : \mathit{Model};\ u : \mathbb{W} \uparrow n \to \mathbb{W} \\ &\qquad \mid \forall\ w_1,\ ...,\ w_n : \mathbb{W} \bullet \exists\ w : \mathbb{W} \bullet \\ &\qquad\quad u\ (w_1,\ ...,\ w_n) \in w \\ &\qquad\quad \land (M \oplus \{i_1 \mapsto w_1,\ ...,\ i_n \mapsto w_n\} \cup \{i_1\ \mathit{decor}\ \spadesuit \mapsto w_1,\ ...,\ i_n\ \mathit{decor}\ \spadesuit \mapsto w_n\}) \mapsto w \in [\![\ e\ ]\!]^{\mathcal{E}} \\ &\qquad \bullet M \mapsto M \cup \lambda\ y : \{j_1,\ ...,\ j_m\} \bullet \lambda\ x : \mathbb{W} \uparrow n \bullet u\ x\ y\}\end{aligned}$$

Given a model M and generic argument sets w_1, ..., w_n, the semantic value of the schema e in that model overridden by the association of the generic parameter names with those sets is w. All combinations of generic argument sets are considered. The function u maps the generic argument sets to a binding in the schema w. The paragraph relates the model M to that model extended with the binding that associates the names of the schema e (namely j_1, ..., j_m) with the corresponding value in the binding resulting from application of u to arbitrary instantiating sets x. Associations for names decorated with the reserved stroke $\spadesuit$ are also introduced whilst determining the semantic value of e, so that references to them from generic types (15.2.6.2) can avoid being captured.

15.2.3.4 Conjecture paragraph

The conjecture paragraph $\vdash?\ p$ END expresses a property that may logically follow from the specification. It may be a starting point for a proof.

$$[\![\ \vdash?\ p\ \texttt{END}\ ]\!]^{\mathcal{D}} \ = \ \mathit{id}\ \mathit{Model}$$

It relates a model to itself: the truth of p in a model does not affect the meaning of the specification.

15.2.3.5 Generic conjecture paragraph

The generic conjecture paragraph $[i_1, ..., i_n] \vdash ?\ p$ `END` expresses a generic property that may logically follow from the specification. It may be a starting point for a proof.

$$[\![\ [i_1, ..., i_n] \vdash ?\ p\ \texttt{END} \ ]\!]^{\mathcal{D}} \;=\; id\ Model$$

It relates a model to itself: the truth of p in a model does not affect the meaning of the specification.

15.2.4 Predicate

The set of models defining the meaning of a predicate is determined from the values of its constituent expressions. The set therefore depends on the particular treatment of undefinedness.

15.2.4.1 Membership predicate

The membership predicate $e_1 \in e_2$ is *true* if and only if the value of e_1 is in the set that is the value of e_2.

$$[\![\ e_1 \in e_2 \ ]\!]^{\mathcal{P}} \;=\; \{M : Model \mid [\![\ e_1 \ ]\!]^{\mathcal{E}} M \in [\![\ e_2 \ ]\!]^{\mathcal{E}} M \bullet M\}$$

In terms of the semantic universe, it is *true* in those models in which the semantic value of e_1 is in the semantic value of e_2, and is *false* otherwise.

15.2.4.2 Truth predicate

A truth predicate is always *true*.

$$[\![\ \mathrm{true} \ ]\!]^{\mathcal{P}} \;=\; Model$$

In terms of the semantic universe, it is *true* in all models.

15.2.4.3 Negation predicate

The negation predicate $\neg\, p$ is *true* if and only if p is *false*.

$$[\![\ \neg\, p \ ]\!]^{\mathcal{P}} \;=\; Model \setminus [\![\ p \ ]\!]^{\mathcal{P}}$$

In terms of the semantic universe, it is *true* in all models except those in which p is *true*.

15.2.4.4 Conjunction predicate

The conjunction predicate $p_1 \wedge p_2$ is *true* if and only if p_1 and p_2 are *true*.

$$[\![\ p_1 \wedge p_2 \ ]\!]^{\mathcal{P}} \;=\; [\![\ p_1 \ ]\!]^{\mathcal{P}} \cap [\![\ p_2 \ ]\!]^{\mathcal{P}}$$

In terms of the semantic universe, it is *true* in those models in which both p_1 and p_2 are *true*, and is *false* otherwise.

15.2.4.5 Universal quantification predicate

The universal quantification predicate $\forall\ e \bullet p$ is *true* if and only if predicate p is *true* for all bindings of the schema e.

$$[\![\ \forall\ e \bullet p \ ]\!]^{\mathcal{P}} \;=\; \{M : Model \mid \forall\ t : [\![\ e \ ]\!]^{\mathcal{E}} M \bullet M \oplus t \in [\![\ p \ ]\!]^{\mathcal{P}} \bullet M\}$$

In terms of the semantic universe, it is *true* in those models for which p is *true* in that model overridden by all bindings in the semantic value of e, and is *false* otherwise.

15.2.5 Expression

Every expression has a semantic value, specified by the following semantic relations. The value of an undefined definite description expression is left loose, and hence the values of larger expressions containing undefined values are also loosely specified.

15.2.5.1 Reference expression

The value of the reference expression that refers to a non-generic definition i is the value of the declaration to which it refers.

$$[\![\ i\ ]\!]^{\mathcal{E}} \ = \ \lambda\ M : Model \bullet M\ i$$

In terms of the semantic universe, its semantic value, given a model M, is that associated with the name i in M.

15.2.5.2 Generic instantiation expression

The value of the generic instantiation expression $i\ [e_1, ..., e_n]$ is a particular instance of the generic referred to by name i.

$$[\![\ i\ [e_1, ..., e_n]\ ]\!]^{\mathcal{E}} \ = \ \lambda\ M : Model \bullet M\ i\ ([\![\ e_1\ ]\!]^{\mathcal{E}} M,\ ...,\ [\![\ e_n\ ]\!]^{\mathcal{E}} M)$$

In terms of the semantic universe, its semantic value, given a model M, is the generic value associated with the name i in M instantiated with the semantic values of the instantiation expressions in M.

15.2.5.3 Set extension expression

The value of the set extension expression $\{\ e_1, ..., e_n\}$ is the set containing the values of its expressions.

$$[\![\ \{\ e_1, ..., e_n\}\ ]\!]^{\mathcal{E}} \ = \ \lambda\ M : Model \bullet \{[\![\ e_1\ ]\!]^{\mathcal{E}} M,\ ...,\ [\![\ e_n\ ]\!]^{\mathcal{E}} M\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set whose members are the semantic values of the member expressions in M.

15.2.5.4 Set comprehension expression

The value of the set comprehension expression $\{\ e_1 \bullet e_2\}$ is the set of values of e_2 for all bindings of the schema e_1.

$$[\![\ \{\ e_1 \bullet e_2\}\ ]\!]^{\mathcal{E}} \ = \ \lambda\ M : Model \bullet \{t_1 : [\![\ e_1\ ]\!]^{\mathcal{E}} M \bullet [\![\ e_2\ ]\!]^{\mathcal{E}}\ (M \oplus t_1)\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of values of e_2 in M overridden with a binding value of e_1 in M.

15.2.5.5 Powerset expression

The value of the powerset expression $\mathbb{P}\, e$ is the set of all subsets of the set that is the value of e.

$$[\![\ \mathbb{P}\, e\ ]\!]^{\mathcal{E}} \ = \ \lambda\ M : Model \bullet \mathbb{P}\,([\![\ e\ ]\!]^{\mathcal{E}} M)$$

In terms of the semantic universe, its semantic value, given a model M, is the powerset of values of e in M.

15.2.5.6 Tuple extension expression

The value of the tuple extension expression $(e_1, ..., e_n)$ is the tuple containing the values of its expressions in order.

$$[\![\ (e_1, ..., e_n)\ ]\!]^{\mathcal{E}} \ = \ \lambda\ M : Model \bullet ([\![\ e_1\ ]\!]^{\mathcal{E}} M,\ ...,\ [\![\ e_n\ ]\!]^{\mathcal{E}} M)$$

In terms of the semantic universe, its semantic value, given a model M, is the tuple whose components are the semantic values of the component expressions in M.

15.2.5.7 Binding extension expression

The value of the binding extension expression $\langle\!|\ i_1 == e_1, ..., i_n == e_n\ |\!\rangle$ is the binding whose names are as enumerated and whose values are those of the associated expressions.

$$[\![\ \langle\!|\ i_1 == e_1, ..., i_n == e_n\ |\!\rangle\ ]\!]^{\mathcal{E}} \ = \ \lambda\ M : Model \bullet \{i_1 \mapsto [\![\ e_1\ ]\!]^{\mathcal{E}} M,\ ...,\ i_n \mapsto [\![\ e_n\ ]\!]^{\mathcal{E}} M\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of pairs enumerated by its names each associated with the semantic value of the associated expression in M.

15.2.5.8 Definite description expression

The value of the definite description expression $\mu\ e_1 \bullet e_2$ is the sole value of e_2 that arises whichever binding is chosen from the set that is the value of schema e_1.

$$\begin{array}{ll} \{M : Model;\ t_1 : \mathbb{W} & \\ \quad \mid t_1 \in [\![\ e_1\ ]\!]^{\mathcal{E}} M & \\ \quad \land (\forall\ t_3 : [\![\ e_1\ ]\!]^{\mathcal{E}} M \bullet [\![\ e_2\ ]\!]^{\mathcal{E}}\ (M \oplus t_3) = [\![\ e_2\ ]\!]^{\mathcal{E}}\ (M \oplus t_1)) & \\ \bullet\ M \mapsto [\![\ e_2\ ]\!]^{\mathcal{E}}\ (M \oplus t_1)\} & \subseteq\ [\![\ \mu\ e_1 \bullet e_2\ ]\!]^{\mathcal{E}} \end{array}$$

In terms of the semantic universe, its semantic value, given a model M in which the value of e_2 in that model overridden by a binding of the schema e_1 is the same regardless of which binding is chosen, is that value of e_2. In other models, it has a semantic value, but this loose definition of the semantics does not say what it is.

15.2.5.9 Variable construction expression

The value of the variable construction expression $[i : e]$ is the set of all bindings whose sole name is i and whose associated value is in the set that is the value of e.

$$[\![\ [i : e]\ ]\!]^{\mathcal{E}} \ = \ \lambda\ M : Model \bullet \{w : [\![\ e\ ]\!]^{\mathcal{E}} M \bullet \{i \mapsto w\}\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of all singleton bindings (sets of pairs) of the name i associated with a value from the set that is the semantic value of e in M.

15.2.5.10 Schema construction expression

The value of the schema construction expression $[e \mid p]$ is the set of all bindings of schema e that satisfy the constraints of predicate p.

$$[\![\ [e \mid p]\ ]\!]^{\mathcal{E}} \ = \ \lambda\ M : Model \bullet \{t : [\![\ e\ ]\!]^{\mathcal{E}} M \mid M \oplus t \in [\![\ p\ ]\!]^{\mathcal{P}} \bullet t\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of the bindings (sets of pairs) that are members of the semantic value of schema e in M such that p is *true* in the model M overridden with that binding.

15.2.5.11 Schema negation expression

The value of the schema negation expression $\neg\ e$ is that set of bindings that are of the same type as those in schema e but that are not in schema e.

$$[\![\ \neg\ e\ ⨾\ \mathbb{P}\,\tau\ ]\!]^{\mathcal{E}} \ = \ \lambda\ M : Model \bullet \{t : [\![\ \tau\ ]\!]^{\mathcal{T}} M \mid \neg\ t \in [\![\ e\ ]\!]^{\mathcal{E}} M \bullet t\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of the bindings (sets of pairs) that are members of the semantic value of the carrier set of schema e in M such that those bindings are not members of the semantic value of schema e in M.

15.2.5.12 Schema conjunction expression

The value of the schema conjunction expression $e_1 \land e_2$ is the schema resulting from merging the signatures of schemas e_1 and e_2 and conjoining their constraints.

$$[\![\ e_1 \land e_2\ ⨾\ \mathbb{P}\,\tau\ ]\!]^{\mathcal{E}} \ = \ \lambda\ M : Model \bullet \{t : [\![\ \tau\ ]\!]^{\mathcal{T}} M;\ t_1 : [\![\ e_1\ ]\!]^{\mathcal{E}} M;\ t_2 : [\![\ e_2\ ]\!]^{\mathcal{E}} M \mid t_1 \cup t_2 = t \bullet t\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of the unions of the bindings (sets of pairs) in the semantic values of e_1 and e_2 in M.

15.2.5.13 Schema universal quantification expression

The value of the schema universal quantification expression $\forall\ e_1 \bullet e_2$ is the set of bindings of schema e_2 restricted to exclude names that are in the signature of e_1, for all bindings of the schema e_1.

$$[\![\ \forall\ e_1 \bullet e_2\ ⨾\ \mathbb{P}\,\tau\ ]\!]^{\mathcal{E}} \ = \ \lambda\ M : Model \bullet \{t_2 : [\![\ \tau\ ]\!]^{\mathcal{T}} M \mid \forall\ t_1 : [\![\ e_1\ ]\!]^{\mathcal{E}} M \bullet t_1 \cup t_2 \in [\![\ e_2\ ]\!]^{\mathcal{E}}\ (M \oplus t_1) \bullet t_2\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of the bindings (sets of pairs) in the semantic values of the carrier set of the type of the entire schema universal quantification expression in M, for which the union of the bindings (sets of pairs) in e_1 and in the whole expression is in the set that is the semantic value of e_2 in the model M overridden with the binding in e_1.

15.2.5.14 Schema renaming expression

The value of the schema renaming expression $e\ [j_1 \ / \ i_1, ..., j_n \ / \ i_n]$ is that schema whose bindings are like those of schema e except that some of its names have been replaced by new names, possibly merging components.

$$\begin{aligned} [\![\ e\ [j_1 \ / \ i_1, ..., j_n \ / \ i_n]\ ]\!]^{\mathcal{E}} \quad = \quad & \lambda\ M : Model \bullet \\ & \{t_1 : [\![\ e\ ]\!]^{\mathcal{E}} M;\ t_2 : \mathbb{W} \mid \\ & \quad t_2 = \{j_1 \mapsto i_1, ..., j_n \mapsto i_n\} \mathbin{⨾} t_1 \cup \{i_1, ..., i_n\} \lhd t_1 \\ & \quad \land\ t_2 \in (\_ \mathbin{⇸} \_) \\ & \bullet\ t_2\} \end{aligned}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of the bindings (sets of pairs) in the semantic value of e in M with the new names replacing corresponding old names. Where components are merged by the renaming, those components shall have the same value.

15.2.6 Type

The value of a type is its carrier set.

NOTE 1 For an expression e with a defined value, $[\![\ e \mathbin{⨾} \tau\ ]\!]^{\mathcal{E}} \in [\![\ \tau\ ]\!]^{\mathcal{T}}$.

NOTE 2 Variable types do not appear in the type annotations of well-typed specifications, so do not need to be given semantics here.

NOTE 3 The value of a generic type, $[\![\ [i_1, ..., i_n]\ \tau\ ]\!]^{\mathcal{T}}$, is never needed, and so is not defined.

NOTE 4 $[\![\ \tau\ ]\!]^{\mathcal{T}} M$ differs from *carrier* τ in that the former application returns a semantic value whereas the latter application returns an annotated parse tree.

15.2.6.1 Given type

$$[\![\ \mathtt{GIVEN}\ i\ ]\!]^{\mathcal{T}} \quad = \quad \lambda\ M : Model \bullet M\ (i\ decor\ \heartsuit)$$

The semantic value of the given type GIVEN i, given a model M, is the semantic value associated with the given type name i in M.

15.2.6.2 Generic parameter type

$$[\![\ \mathtt{GENTYPE}\ i\ ]\!]^{\mathcal{T}} \quad = \quad \lambda\ M : Model \bullet M\ (i\ decor\ \spadesuit)$$

The semantic value of the generic type GENTYPE i, given a model M, is the semantic value associated with generic parameter name i in M.

15.2.6.3 Powerset type

$$[\![\ \mathbb{P}\,\tau\ ]\!]^{\mathcal{T}} \quad = \quad \lambda\ M : Model \bullet \mathbb{P}\,([\![\ \tau\ ]\!]^{\mathcal{T}} M)$$

The semantic value of the set type $\mathbb{P}\,\tau$, given a model M, is the powerset of the semantic value of type τ in M.

15.2.6.4 Cartesian product type

$$[\![\ \tau_1 \times ... \times \tau_n\ ]\!]^{\mathcal{T}} \quad = \quad \lambda\ M : \mathit{Model} \bullet ([\![\ \tau_1\ ]\!]^{\mathcal{T}} M) \times ... \times ([\![\ \tau_n\ ]\!]^{\mathcal{T}} M)$$

The semantic value of the Cartesian product type $\tau_1 \times ... \times \tau_n$, given a model M, is the Cartesian product of the semantic values of types $\tau_1 ... \tau_n$ in M.

15.2.6.5 Schema type

$$\begin{aligned} [\![\ [i_1 : \tau_1;\ ...;\ i_n : \tau_n]\ ]\!]^{\mathcal{T}} \quad = \quad & \lambda\ M : \mathit{Model} \\ & \bullet \{t : \{i_1,\ ...,\ i_n\} \rightarrow \mathbb{W} \mid t\ i_1 \in [\![\ \tau_1\ ]\!]^{\mathcal{T}} M \wedge ... \wedge t\ i_n \in [\![\ \tau_n\ ]\!]^{\mathcal{T}} M \bullet t\} \end{aligned}$$

The semantic value of the schema type $[i_1 : \tau_1;\ ...;\ i_n : \tau_n]$, given a model M, is the set of bindings, represented by sets of pairs of names and values, for which the names are those of the schema type and the associated values are the semantic values of the corresponding types in M.

Annex A
(normative)

Mark-ups

A.1 Introduction

A mark-up is a mapping to (or from) the ISO/IEC 10646 representation. The ISO/IEC 10646 representation may be used directly—the identity function is an acceptable mapping. However, not all systems support ISO/IEC 10646, the definitive representation of Z characters (clause 6). This annex defines two mark-ups based on 7-bit ASCII [4]:

- a LaTeX [9] mark-up, suitable for processing by that tool to render Z characters in their mathematical form;
- an email, or lightweight ASCII, mark-up, suitable for rendering Z characters on a low resolution device, such as an ASCII-character-based terminal, or in email correspondence.

The mark-ups described in this annex show how to translate between a 'mark-up token' (string of ASCII mark-up characters) into the corresponding string of Z characters. Remaining individual mark-up characters that do not form a special mark-up token (such as digits, Latin letters, and much punctuation) are converted directly to the corresponding Z character, e.g. from ASCII-*xy* to `0000 00`*xy* in ISO/IEC 10646. Use of different sequences of mark-up characters that correspond to the same Z characters are permitted by this International Standard, as the same tokens will result. Tools may require tokens to be marked-up consistently.

A chosen mark-up language may also be used to specify a particular rendering for the characters, for example, bold or italic.

The semantics of a specification are specified by the normative clauses of this International Standard on the assumption that its sections are in a definition before use order. This restriction need not be imposed on the order in which sections are presented to a human reader or to a tool. If the mark-up of a specification is scanned recognising only section headers, then a definition before use order for the sections can be determined (unless there are erroneous cycles in the parents relation), and the contents of the sections can then be scanned in that order.

A.2 LaTeX mark-up

A LaTeX command is a backslash '\' followed by a string of alphabetic characters (up to the first non-alphabetic character), or by a single non-alphabetic character.

A.2.1 Letter characters

A.2.1.1 Greek alphabet characters

Only the minimal subset of Greek alphabet defined in 6.2 need be supported by an implementation. LaTeX does not support upper case Greek letters that look like Roman counterparts. Those Greek characters that are supported shall use the mark-up given here.

LaTeX command	Z character string
`\Delta`	Δ
`\Xi`	Ξ
`\theta`	θ
`\lambda`	λ
`\mu`	μ

A.2.1.2 Other Z core language letter characters

LaTeX command	Z character string
`\arithmos`	$\mathbb{A}$
`\nat`	$\mathbb{N}$
`\power`	$\mathbb{P}$

A.2.2 Special characters

A.2.2.1 Special characters except Box characters

LaTeX command	Z character string
`\_`	_
`\{`	{
`\}`	}
`\ldata`	⟪
`\rdata`	⟫
`\lblot`	⦉
`\rblot`	⦊

Subscripts and superscripts shall be marked up as follows:

LaTeX command	Z character string
`^` ⟨single LaTeX token⟩	$\nearrow$⟨ Z string ⟩ $\swarrow$
`^{` ⟨LaTeX tokens⟩ `}`	$\nearrow$⟨ Z string ⟩ $\swarrow$
`_` ⟨single LaTeX token⟩	$\searrow$⟨ Z string ⟩ $\nwarrow$
`_{` ⟨LaTeX tokens⟩ `}`	$\searrow$⟨ Z string ⟩ $\nwarrow$

EXAMPLE LaTeX mark-up `x^1` corresponds to Z character string '$x \nearrow 1 \swarrow$', which may be rendered 'x^1'
LaTeX mark-up `x^{1}` corresponds to Z character string '$x \nearrow 1 \swarrow$', which may be rendered 'x^1'
LaTeX mark-up `x^{\Delta S}` corresponds to Z character string '$x \nearrow \Delta S \swarrow$', which may be rendered '$x^{\Delta S}$'
LaTeX mark-up `\exists_1` corresponds to Z character string '$\exists \searrow 1 \nwarrow$', which may be rendered '$\exists_1$'
LaTeX mark-up `\exists_{1}` corresponds to Z character string '$\exists \searrow 1 \nwarrow$', which may be rendered '$\exists_1$'
LaTeX mark-up `\exists_{\Delta S}` corresponds to Z character string '$\exists \searrow \Delta S \nwarrow$', which may be rendered '$\exists_{\Delta S}$'
LaTeX mark-up `x_a^b` corresponds to Z character string '$x \searrow a \nwarrow \nearrow b \swarrow$', which may be rendered '$x_a{}^b$'
LaTeX mark-up `x_{a^b}` corresponds to Z character string '$x \searrow a \nearrow b \swarrow \nwarrow$', which may be rendered 'x_{a^b}'

A.2.2.2 Box characters

The `ENDCHAR` character is used to mark the end of a `Paragraph`. The `NLCHAR` character is used to mark a hard newline (see 7.5). Different implementations may represent these characters in different ways.

The box characters are described in A.2.7, on paragraph mark-up.

A.2.3 Symbol characters (except mathematical toolkit characters)

LaTeX command	Z character string
`\vdash`	$\vdash$
`\land`	$\land$
`\lor`	$\lor$
`\implies`	$\Rightarrow$
`\iff`	$\Leftrightarrow$
`\lnot`	$\neg$
`\forall`	$\forall$
`\exists`	$\exists$
`\cross`	$\times$
`\in`	$\in$
`@`	$\bullet$
`\hide`	$\setminus$
`\project`	$\upharpoonright$
`\semi`	$\fatsemi$
`\pipe`	$\gg$

A.2.4 Core tokens

The Roman typeface used for core tokens in this International Standard is obtained using the mark-up defined in A.2.10, with the following exceptions where there would otherwise be clashes with LaTeX keywords.

LaTeX command	Z character string
`\IF`	if
`\THEN`	then
`\ELSE`	else
`\LET`	let
`\SECTION`	section

A.2.5 Mathematical toolkit characters and tokens

The mathematical toolkit need not be supported by an implementation. If any of its tokens are supported, they shall use the mark-up given here.

LaTeX command	**Z character string**
`\rel`	↔
`\fun`	→
`\neq`	≠
`\notin`	∉
`\emptyset`	∅
`\subseteq`	⊆
`\subset`	⊂
`\cup`	∪
`\cap`	∩
`\setminus`	∖
`\symdiff`	⊖
`\bigcup`	⋃
`\bigcap`	⋂
`\finset`	𝔽
`\mapsto`	↦
`\comp`	⨟
`\circ`	∘
`\dres`	◁
`\rres`	▷
`\ndres`	⩤
`\nrres`	⩥
`\inv`	∼
`\limg`	⦇
`\rimg`	⦈
`\oplus`	⊕
`\plus`	↗+↙
`\star`	↗∗↙
`\pfun`	⇸
`\pinj`	⤔
`\inj`	↣
`\psurj`	⤀
`\surj`	↠
`\bij`	⤖
`\ffun`	⇻
`\finj`	⤕
`\num`	ℤ
`\negate`	`-`
`-`	−
`\leq`	≤
`<`	<
`\geq`	≥
`>`	>
`\upto`	`..`
`\#`	#
`\langle`	⟨
`\rangle`	⟩
`\cat`	⁀
`\extract`	↿
`\filter`	↾
`\dcat`	⁀/

A.2.6 Section header mark-up

Section headers are enclosed in a LaTeX `zsection` environment. The `\begin{zsection}` is converted to `SPACE`. The `\end{zsection}` is converted to `ENDCHAR`.

```
\begin{zsection}
\SECTION NAME \parents ...
\end{zsection}
```

A.2.7 Paragraph mark-up

Each formal Z paragraph appears between a pair of `\begin{xxx}` and `\end{xxx}` LaTeX commands. Text not appearing between such commands is informal accompanying text.

For boxed paragraphs, the `\begin{xxx}` command indicates some box character, while for other paragraphs the `\begin{xxx}` command is Z whitespace. Any middle line in a boxed paragraph is marked-up using the `\where` LaTeX command, which corresponds to the Z | character. The `\end{xxx}` command represents the Z `ENDCHAR` character.

A.2.7.1 Axiomatic description paragraph mark-up

```
\begin{axdef}
DeclPart
\where
Predicate
\end{axdef}
```

The mark-up `\begin{axdef}` is translated to an `AXCHAR` character. The mark-up `\where` is translated to a | character. The mark-up `\end{axdef}` is translated to an `ENDCHAR` character.

A.2.7.2 Schema definition paragraph mark-up

```
\begin{schema}{NAME}
DeclPart
\where
Predicate
\end{schema}
```

The mark-up `\begin{schema}{    }` is translated to a `SCHCHAR` character. The mark-up `\where` is translated to a | character. The mark-up `\end{schema}` is translated to an `ENDCHAR` character.

A.2.7.3 Generic axiomatic description paragraph mark-up

```
\begin{gendef}[Formals]
DeclPart
\where
Predicate
\end{gendef}
```

The mark-up `\begin{gendef}` is translated to an `AXCHAR` and a `GENCHAR` character. The mark-up `\where` is translated to a | character. The mark-up `\end{gendef}` is translated to an `ENDCHAR` character.

A.2.7.4 Generic schema definition paragraph mark-up

```
\begin{schema}{NAME}[Formals]
DeclPart
\where
Predicate
\end{schema}
```

The mark-up `\begin{schema}{    }` is translated to a `SCHCHAR` and a `GENCHAR` character. The mark-up `\where` is translated to a | character. The mark-up `\end{schema}` is translated to an `ENDCHAR` character.

A.2.7.5 Free types paragraph mark-up

```
\begin{zed}
Freetype, { \& , Freetype }
\end{zed}
```

A.2.7.6 Other paragraph mark-up

All other paragraphs are enclosed in a pair of `\begin{zed}` and `\end{zed}` commands. `\begin{zed}` is converted to white space and `\end{zed}` is converted to `ENDCHAR`.

A.2.8 LaTeX whitespace mark-up

LaTeX has 'hard' white space (explicit LaTeX mark-up) and 'soft' white space (ASCII white space characters such as space, tab, and new line that are converted to the Z character `SPACE`).

The hard white space is converted as follows:

LaTeX command	**Z character string**
`{`	(empty)
`}`	(empty)
`~`	`SPACE`
`\,`	`SPACE`
`\!`	`SPACE`
`\`(space)	`SPACE`
`\;`	`SPACE`
`\:`	`SPACE`
`\t1`	`SPACE`
`\t2`	`SPACE`
`\t3`	`SPACE`
`\t4`	`SPACE`
`\t5`	`SPACE`
`\t6`	`SPACE`
`\t7`	`SPACE`
`\t8`	`SPACE`
`\t9`	`SPACE`
`\\`	`NLCHAR`
`\also`	`NLCHAR`

The conversion of LaTeX Greek characters shall consume any immediately following soft white space. The conversion of LaTeX symbol characters shall preserve any following soft white space. Any remaining soft white space shall be converted to the `SPACE` character.

EXAMPLE 1 The LaTeX command '`\Delta S`' converts to the Z string 'ΔS'.

EXAMPLE 2 The LaTeX command '`\Delta~S`' converts to the Z string '$\Delta\ S$'.

EXAMPLE 3 The LaTeX command '`\power S`' converts to the Z string '$\mathbb{P}\ S$'.

A.2.9 Introducing new Z characters

A new Z character is introduced by the following one-line directive.

```
%%Zchar \LaTeXcommand U+nnnn
```

Subsequent occurrences of this `\LaTeXcommand` shall be converted to the corresponding character in plane 1 of ISO/IEC 10646.

NOTE 1 There may be an accompanying LaTeX `\DeclareMathSymbol` for the same `\LaTeXcommand`, giving a corresponding character code as its expansion. The two characters thus defined are not required to be the same, but if they differ, confusion may ensue.

A LaTeX command to abbreviate the LaTeX mark-up of a sequence of Z characters is introduced by the following one-line directive.

```
%%Zstring \LaTeXcommand Zstring
```

The `Zstring` excludes any leading spaces.

NOTE 2 There will typically be an accompanying LaTeX `\newcommand` for the same `\LaTeXcommand`, giving the same `Zstring` as its expansion. The two expansions are not required to be the same, but if they differ, confusion may ensue.

Subsequent occurrences of this `\LaTeXcommand` shall be replaced by the corresponding `Zstring` and that mark-up shall be converted.

The scope of one of these directives is the rest of the section in which it appears and any sections of which it is an ancestor, excluding the headers of those sections.

NOTE 3 This is the same as the scope of a global definition.

EXAMPLE `%%Zchar \sqsubseteq U+2291`
`%%Zstring \nattwo \nat_2`

A.2.10 Remaining LaTeX mark-up

Any remaining LaTeX command names enclosed in braces, '`{\aToken}`', shall be converted to the equivalent Z character string with the braces and leading backslash removed, as '`aToken`'.

Any remaining LaTeX command names shall be converted to the equivalent Z character string with the leading backslash removed, and with a `SPACE` character added at the beginning and end, as '` aToken `'.

EXAMPLE 1 LaTeX mark-up: '`{\dom}s`', Z character string: '`doms`'.
LaTeX mark-up: '`\dom s`', Z character string: '` dom  s`'.

EXAMPLE 2 LaTeX mark-up: `\IF \disjoint a \THEN x = y \mod z \ELSE x = y \div z`
Z character string: `if disjoint a then x = y mod z else x = y div z`
A possible rendering: if *disjoint* *a* then $x = y$ *mod* z else $x = y$ *div* z

A.3 Email mark-up

This email mark-up is designed primarily as a human-readable lightweight mark-up, but may also be processed by tools. The character '`%`' delimits an ASCII string used to represent a Z character string, for example '$\times$' as '`%x%`'. This disambiguates it from, for example, the name '`x`'.

Where there is no danger of ambiguity (for the human reader) the trailing '`%`' character, or both '`%`' characters, may be omitted to reduce clutter.

A literal '`%`' character may be introduced into the text as '`%%`'.

A.3.1 Letter characters

In the following, the email string is to be used surrounded by a leading and trailing '`%`' character.

Names that use only ASCII characters, or that are composed out of previously defined Z characters, are not listed here.

A.3.1.1 Greek alphabet characters

Only the minimal subset of Greek alphabet defined in 6.2 need be supported by an implementation. Those Greek characters that are supported shall use the mark-up given here.

Email string	**Z character string**
`Delta`	Δ
`Xi`	Ξ
`theta`	θ
`lambda`	λ
`mu`	μ

A.3.1.2 Other Z core language letter characters

Email string	**Z character string**
`arithmos`	$\mathbb{A}$
`N`	$\mathbb{N}$
`P`	$\mathbb{P}$

A.3.2 Special characters

A.3.2.1 Special characters except Box characters

Email string	**Z character string**
`/^`	$\nearrow$
`v/`	$\swarrow$
`\v`	$\searrow$
`^\`	$\nwarrow$
`<<`	$\langle\!\langle$
`>>`	$\rangle\!\rangle$
`<|`	$\lhd$
`|>`	$\rhd$

A.3.2.2 Box characters

The `ENDCHAR` character is used to mark the end of a `Paragraph`. There are several different mark-ups for the `ENDCHAR` character, depending on the kind of paragraph. They are described in A.3.5, on paragraph mark-up.

The `NLCHAR` character is used to mark a hard newline (see 7.5).

The email form of the box characters mimicks the mathematical form, as various boxes drawn around the text. They are described in A.3.5, on paragraph mark-up.

A.3.3 Symbol characters (except mathematical toolkit characters)

<table>
<tr><th>Email string</th><th>Z character string</th></tr>
<tr><td>|</td><td>$|$</td></tr>
<tr><td>|-</td><td>$\vdash$</td></tr>
<tr><td>/\</td><td>$\land$</td></tr>
<tr><td>\/</td><td>$\lor$</td></tr>
<tr><td>==></td><td>$\Rightarrow$</td></tr>
<tr><td><=></td><td>$\Leftrightarrow$</td></tr>
<tr><td>not</td><td>$\lnot$</td></tr>
<tr><td>A</td><td>$\forall$</td></tr>
<tr><td>E</td><td>$\exists$</td></tr>
<tr><td>x</td><td>$\times$</td></tr>
<tr><td>e</td><td>$\in$</td></tr>
<tr><td>@</td><td>$\bullet$</td></tr>
<tr><td>S\</td><td>$\setminus$</td></tr>
<tr><td>S|\</td><td>$\upharpoonright$</td></tr>
<tr><td>S;</td><td>$\fatsemi$</td></tr>
<tr><td>S>></td><td>$\gg$</td></tr>
</table>

A.3.4 Mathematical toolkit characters and tokens

The mathematical toolkit need not be supported by an implementation. If any of its tokens are supported, they shall use the mark-up given here.

<table>
<tr><th>Email string</th><th>Z character string</th></tr>
<tr><td><code><--></code></td><td>↔</td></tr>
<tr><td><code>--></code></td><td>→</td></tr>
<tr><td><code>/=</code></td><td>≠</td></tr>
<tr><td><code>/e</code></td><td>∉</td></tr>
<tr><td><code>(/)</code></td><td>∅</td></tr>
<tr><td><code>c_</code></td><td>⊆</td></tr>
<tr><td><code>c</code></td><td>⊂</td></tr>
<tr><td><code>u</code></td><td>∪</td></tr>
<tr><td><code>n</code></td><td>∩</td></tr>
<tr><td><code>(-)</code></td><td>⊖</td></tr>
<tr><td><code>uu</code></td><td>⋃</td></tr>
<tr><td><code>nn</code></td><td>⋂</td></tr>
<tr><td><code>F</code></td><td>𝔽</td></tr>
<tr><td><code>|--></code></td><td>↦</td></tr>
<tr><td><code>;</code></td><td>⨾</td></tr>
<tr><td><code>o</code></td><td>∘</td></tr>
<tr><td><code><:</code></td><td>◁</td></tr>
<tr><td><code>:></code></td><td>▷</td></tr>
<tr><td><code><-:</code></td><td>⩤</td></tr>
<tr><td><code>:-></code></td><td>⩥</td></tr>
<tr><td><code>(|</code></td><td>⦇</td></tr>
<tr><td><code>|)</code></td><td>⦈</td></tr>
<tr><td><code>(+)</code></td><td>⊕</td></tr>
<tr><td><code>+</code></td><td>⁺</td></tr>
<tr><td><code>*</code></td><td>*</td></tr>
<tr><td><code>-|-></code></td><td>⇸</td></tr>
<tr><td><code>>-|-></code></td><td>⤔</td></tr>
<tr><td><code>>--></code></td><td>↣</td></tr>
<tr><td><code>-|->></code></td><td>⤀</td></tr>
<tr><td><code>-->></code></td><td>↠</td></tr>
<tr><td><code>>-->></code></td><td>⤖</td></tr>
<tr><td><code>-||-></code></td><td>⇻</td></tr>
<tr><td><code>>-||-></code></td><td>⤕</td></tr>
<tr><td><code>Z</code></td><td>ℤ</td></tr>
<tr><td><code>-</code></td><td>- (unary negation)</td></tr>
<tr><td><code><=</code></td><td>≤</td></tr>
<tr><td><code>>=</code></td><td>≥</td></tr>
<tr><td><code><</code></td><td>⟨</td></tr>
<tr><td><code>></code></td><td>⟩</td></tr>
<tr><td><code>^</code></td><td>⁀</td></tr>
<tr><td><code>/|</code></td><td>↿</td></tr>
<tr><td><code>|\</code></td><td>↾</td></tr>
</table>

A.3.5 Paragraph mark-up

A.3.5.1 Axiomatic description paragraph mark-up

```
+..
DeclPart
|--
Predicate
```



```
-..
```

The mark-up `+..` is translated to an `AXCHAR` character. The mark-up `|--` is translated to a | character. The mark-up `-..` is translated to an `ENDCHAR` character.

A.3.5.2 Schema definition paragraph mark-up

```
+-- NAME ---
DeclPart
|--
Predicate
---
```

The mark-up `+--    ---` is translated to a `SCHCHAR` character. The mark-up `|--` is translated to a | character. The mark-up `---` is translated to an `ENDCHAR` character.

A.3.5.3 Generic axiomatic description paragraph mark-up

```
+== [Formals] ===
DeclPart
|--
Predicate
-==
```

The mark-up `+==    ===` is translated to an `AXCHAR` and a `GENCHAR` character. The mark-up `|--` is translated to a | character. The mark-up `-==` is translated to an `ENDCHAR` character.

A.3.5.4 Generic schema definition paragraph mark-up

```
+-- NAME[Formals] ---
DeclPart
|--
Predicate
---
```

The mark-up `+--    ---` is translated to a `SCHCHAR` and a `GENCHAR` character. The mark-up `|--` is translated to a | character. The mark-up `---` is translated to an `ENDCHAR` character.

A.3.5.5 Other paragraph mark-up

Unboxed formal paragraphs (and informal paragraphs where necessary) end with a '`%%`' on a line by itself, which mark-up is translated to an `ENDCHAR` character.

Annex B
(normative)
Mathematical toolkit

B.1 Introduction

The mathematical toolkit is an optional extension to the compulsory core language. It comprises a hierarchy of related sections, each defining operators that are widely used in common application domains.

Figure B.1 – Parent relation between sections of the mathematical toolkit

$$\begin{tikzcd}
\text{set_toolkit} & & \\
\text{relation_toolkit} \arrow[u] & & \\
\text{function_toolkit} \arrow[u] & & \text{number_toolkit} \\
 & \text{sequence_toolkit} \arrow[ul] \arrow[ur] & \\
 & \text{standard_toolkit} \arrow[u] &
\end{tikzcd}$$

The division of the mathematical toolkit into separate sections allows use of certain subsets of the toolkit rather than its entirety. For example, if sequences are not used in a particular specification, then using *function_toolkit* and *number_toolkit* as parents avoids bringing the notation of *sequence_toolkit* into scope. Notations that are not reused can be given different definitions.

A specification without a section header has section *standard_toolkit* as an implicit parent.

B.2 Preliminary definitions

section *set_toolkit*

B.2.1 Relations

generic 5 rightassoc $(\_ \leftrightarrow \_)$

$$X \leftrightarrow Y == \mathbb{P}(X \times Y)$$

$X \leftrightarrow Y$ is the set of relations between X and Y, that is, the set of all sets of ordered pairs whose first members are members of X and whose second members are members of Y.

B.2.2 Total functions

generic 5 rightassoc $(\_ \rightarrow \_)$

$$X \rightarrow Y == \{\, f : X \leftrightarrow Y \mid \forall x : X \bullet \exists_1\, y : Y \bullet (x, y) \in f \,\}$$

$X \rightarrow Y$ is the set of all total functions from X to Y, that is, the set of all relations between X and Y such that each x in X is related to exactly one y in Y.

B.3 Sets

B.3.1 Inequality relation

relation (_ ≠ _)

$$[X]$$
$$\_ \neq \_ : X \leftrightarrow X$$
$$\forall x, y : X \bullet x \neq y \Leftrightarrow \neg\, x = y$$

Inequality is the relation between those values of the same type that are not equal to each other.

B.3.2 Non-membership

relation (_ ∉ _)

$$[X]$$
$$\_ \notin \_ : X \leftrightarrow \mathbb{P}\, X$$
$$\forall x : X;\ a : \mathbb{P}\, X \bullet x \notin a \Leftrightarrow \neg\, x \in a$$

Non-membership is the relation between those values of a type, x, and sets of values of that type, a, for which x is not a member of a.

B.3.3 Empty set

$$\varnothing[X] == \{\ x : X \mid \mathit{false}\ \}$$

The empty set of any type is the set of that type that has no members.

B.3.4 Subset relation

relation (_ ⊆ _)

$$[X]$$
$$\_ \subseteq \_ : \mathbb{P}\, X \leftrightarrow \mathbb{P}\, X$$
$$\forall a, b : \mathbb{P}\, X \bullet a \subseteq b \Leftrightarrow (\forall x : a \bullet x \in b)$$

Subset is the relation between two sets of the same type, a and b, such that every member of a is a member of b.

B.3.5 Proper subset relation

relation (_ ⊂ _)

$$[X]$$
$$\_ \subset \_ : \mathbb{P}\, X \leftrightarrow \mathbb{P}\, X$$
$$\forall a, b : \mathbb{P}\, X \bullet a \subset b \Leftrightarrow a \subseteq b \land a \neq b$$

Proper subset is the relation between two sets of the same type, a and b, such that a is a subset of b, and a and b are not equal.

B.3.6 Non-empty subsets

$$\mathbb{P}_1\, X == \{\, a : \mathbb{P}\, X \mid a \neq \varnothing \,\}$$

If X is a set, then $\mathbb{P}_1\, X$ is the set of all non-empty subsets of X.

NOTE The word $\mathbb{P}$ is established as a generic operator by the prelude.

B.3.7 Set union

function 30 leftassoc $(\_ \cup \_)$

$$[X]$$
$$\_ \cup \_ : \mathbb{P}\, X \times \mathbb{P}\, X \to \mathbb{P}\, X$$
$$\forall\, a, b : \mathbb{P}\, X \bullet a \cup b = \{\, x : X \mid x \in a \lor x \in b \,\}$$

The union of two sets of the same type is the set of values that are members of either set.

B.3.8 Set intersection

function 40 leftassoc $(\_ \cap \_)$

$$[X]$$
$$\_ \cap \_ : \mathbb{P}\, X \times \mathbb{P}\, X \to \mathbb{P}\, X$$
$$\forall\, a, b : \mathbb{P}\, X \bullet a \cap b = \{\, x : X \mid x \in a \land x \in b \,\}$$

The intersection of two sets of the same type is the set of values that are members of both sets.

B.3.9 Set difference

function 30 leftassoc $(\_ \setminus \_)$

$$[X]$$
$$\_ \setminus \_ : \mathbb{P}\, X \times \mathbb{P}\, X \to \mathbb{P}\, X$$
$$\forall\, a, b : \mathbb{P}\, X \bullet a \setminus b = \{\, x : X \mid x \in a \land x \notin b \,\}$$

The difference of two sets of the same type is the set of values that are members of the first set but not members of the second set.

B.3.10 Set symmetric difference

function 25 leftassoc $(\_ \ominus \_)$

$$[X]$$
$$\_ \ominus \_ : \mathbb{P}\, X \times \mathbb{P}\, X \to \mathbb{P}\, X$$
$$\forall\, a, b : \mathbb{P}\, X \bullet a \ominus b = \{\, x : X \mid \lnot\, (x \in a \Leftrightarrow x \in b) \,\}$$

The symmetric set difference of two sets of the same type is the set of values that are members of one set, or the other, but not members of both.

B.3.11 Generalized union

$$[X]$$
$$\bigcup : \mathbb{P}\,\mathbb{P}\,X \rightarrow \mathbb{P}\,X$$
$$\forall A : \mathbb{P}\,\mathbb{P}\,X \bullet \bigcup A = \{\, x : X \mid \exists\, a : A \bullet x \in a \,\}$$

The generalized union of a set of sets of the same type is the set of values of that type that are members of at least one of the sets.

B.3.12 Generalized intersection

$$[X]$$
$$\bigcap : \mathbb{P}\,\mathbb{P}\,X \rightarrow \mathbb{P}\,X$$
$$\forall A : \mathbb{P}\,\mathbb{P}\,X \bullet \bigcap A = \{\, x : X \mid \forall\, a : A \bullet x \in a \,\}$$

The generalized intersection of a set of sets of values of the same type is the set of values of that type that are members of every one of the sets.

B.4 Finite sets

B.4.1 Finite subsets

generic 80 ($\mathbb{F}\,\_$)

$$\mathbb{F}\,X == \bigcap\{\, A : \mathbb{P}\,\mathbb{P}\,X \mid \varnothing \in A \land (\forall\, a : A;\ x : X \bullet a \cup \{x\} \in A) \,\}$$

If X is a set, then $\mathbb{F}\,X$ is the set of all finite subsets of X. The set of finite subsets of X is the smallest set that contains the empty set and is closed under the action of adding single elements of X.

B.4.2 Non-empty finite subsets

$$\mathbb{F}_1\,X == \mathbb{F}\,X \setminus \{\varnothing\}$$

If X is a set, then $\mathbb{F}_1\,X$ is the set of all non-empty finite subsets of X. The set of non-empty finite subsets of X is the smallest set that contains the singleton sets of X and is closed under the action of adding single elements of X.

B.5 More notations for relations

section *relation_toolkit* parents *set_toolkit*

B.5.1 First component projection

$$[X, Y]$$
$$\mathit{first} : X \times Y \rightarrow X$$
$$\forall\, p : X \times Y \bullet \mathit{first}\ p = p.1$$

For any ordered pair p, $\mathit{first}\ p$ is the first component of the pair.

B.5.2 Second component projection

$$[X, Y]$$
$$\mathit{second} : X \times Y \rightarrow Y$$
$$\forall\, p : X \times Y \bullet \mathit{second}\ p = p.2$$

For any ordered pair p, $\mathit{second}\ p$ is the second component of the pair.

B.5.3 Maplet

function 10 leftassoc $(\_ \mapsto \_)$

$$
\begin{array}{l}
[X, Y] \\
\hline
\_ \mapsto \_ : X \times Y \rightarrow X \times Y \\
\hline
\forall x : X; \; y : Y \bullet x \mapsto y = (x, y)
\end{array}
$$

The maplet forms an ordered pair from two values; $x \mapsto y$ is just another notation for (x, y).

B.5.4 Domain

$$
\begin{array}{l}
[X, Y] \\
\hline
dom : (X \leftrightarrow Y) \rightarrow \mathbb{P}\, X \\
\hline
\forall r : X \leftrightarrow Y \bullet dom\, r = \{ \; p : r \bullet p.1 \; \}
\end{array}
$$

The domain of a relation r is the set of first components of the ordered pairs in r.

B.5.5 Range

$$
\begin{array}{l}
[X, Y] \\
\hline
ran : (X \leftrightarrow Y) \rightarrow \mathbb{P}\, Y \\
\hline
\forall r : X \leftrightarrow Y \bullet ran\, r = \{ \; p : r \bullet p.2 \; \}
\end{array}
$$

The range of a relation r is the set of second components of the ordered pairs in r.

B.5.6 Identity relation

generic 80 $(\; id \, \_ \;)$

$id\, X == \{ \; x : X \bullet x \mapsto x \; \}$

The identity relation on a set X is the relation that relates every member of X to itself.

B.5.7 Relational composition

function 40 leftassoc $(\_ \fatsemi \_)$

$$
\begin{array}{l}
[X, Y, Z] \\
\hline
\_ \fatsemi \_ : (X \leftrightarrow Y) \times (Y \leftrightarrow Z) \rightarrow (X \leftrightarrow Z) \\
\hline
\forall r : X \leftrightarrow Y; \; s : Y \leftrightarrow Z \bullet r \fatsemi s = \{ \; p : r; \; q : s \mid p.2 = q.1 \bullet p.1 \mapsto q.2 \; \}
\end{array}
$$

The relational composition of a relation $r : X \leftrightarrow Y$ and $s : Y \leftrightarrow Z$ is a relation of type $X \leftrightarrow Z$ formed by taking all the pairs p of r and q of s, where the second component of p is equal to the first component of q, and relating the first component of p with the second component of q.

B.5.8 Functional composition

function 40 leftassoc $(\_ \circ \_)$

$$
\begin{array}{l}
[X, Y, Z] \\
\hline
\_ \circ \_ : (Y \leftrightarrow Z) \times (X \leftrightarrow Y) \rightarrow (X \leftrightarrow Z) \\
\hline
\forall r : X \leftrightarrow Y; \; s : Y \leftrightarrow Z \bullet s \circ r = r \fatsemi s
\end{array}
$$

The functional composition of s and r is the same as the relational composition of r and s.

B.5.9 Domain restriction

function 65 rightassoc (_ ◁ _)

$$
\begin{array}{l}
[X, Y] \\
\_ \lhd \_ : \mathbb{P}\, X \times (X \leftrightarrow Y) \rightarrow (X \leftrightarrow Y) \\
\hline
\forall\, a : \mathbb{P}\, X;\ r : X \leftrightarrow Y \bullet a \lhd r = \{\ p : r \mid p.1 \in a\ \}
\end{array}
$$

The domain restriction of a relation $r : X \leftrightarrow Y$ by a set $a : \mathbb{P}\, X$ is the set of pairs in r whose first components are in a.

B.5.10 Range restriction

function 60 leftassoc (_ ▷ _)

$$
\begin{array}{l}
[X, Y] \\
\_ \rhd \_ : (X \leftrightarrow Y) \times \mathbb{P}\, Y \rightarrow (X \leftrightarrow Y) \\
\hline
\forall\, r : X \leftrightarrow Y;\ b : \mathbb{P}\, Y \bullet r \rhd b = \{\ p : r \mid p.2 \in b\ \}
\end{array}
$$

The range restriction of a relation $r : X \leftrightarrow Y$ by a set $b : \mathbb{P}\, Y$ is the set of pairs in r whose second components are in b.

B.5.11 Domain subtraction

function 65 rightassoc (_ ⩤ _)

$$
\begin{array}{l}
[X, Y] \\
\_ \ntriangleleft \_ : \mathbb{P}\, X \times (X \leftrightarrow Y) \rightarrow (X \leftrightarrow Y) \\
\hline
\forall\, a : \mathbb{P}\, X;\ r : X \leftrightarrow Y \bullet a \ntriangleleft r = \{\ p : r \mid p.1 \notin a\ \}
\end{array}
$$

The domain subtraction of a relation $r : X \leftrightarrow Y$ by a set $a : \mathbb{P}\, X$ is the set of pairs in r whose first components are not in a.

B.5.12 Range subtraction

function 60 leftassoc (_ ⩥ _)

$$
\begin{array}{l}
[X, Y] \\
\_ \ntriangleright \_ : (X \leftrightarrow Y) \times \mathbb{P}\, Y \rightarrow (X \leftrightarrow Y) \\
\hline
\forall\, r : X \leftrightarrow Y;\ b : \mathbb{P}\, Y \bullet r \ntriangleright b = \{\ p : r \mid p.2 \notin b\ \}
\end{array}
$$

The range subtraction of a relation $r : X \leftrightarrow Y$ by a set $b : \mathbb{P}\, Y$ is the set of pairs in r whose second components are not in b.

B.5.13 Relational inversion

function 90 (_ $^\sim$)

$$
\begin{array}{l}
[X, Y] \\
\_^{\sim} : (X \leftrightarrow Y) \rightarrow (Y \leftrightarrow X) \\
\hline
\forall\, r : X \leftrightarrow Y \bullet r^{\sim} = \{\ p : r \bullet p.2 \mapsto p.1\ \}
\end{array}
$$

The inverse of a relation is the relation obtained by reversing every ordered pair in the relation.

B.5.14 Relational image

function 90 (_ ⦇ _ ⦈)

$$
\begin{array}{l}
[X, Y] \\
\_ (\!| \_ |\!) : (X \leftrightarrow Y) \times \mathbb{P}\, X \to \mathbb{P}\, Y \\
\hline
\forall\, r : X \leftrightarrow Y;\ a : \mathbb{P}\, X \bullet r (\!|\ a\ |\!) = \{\ p : r \mid p.1 \in a \bullet p.2\ \}
\end{array}
$$

The relational image of a set $a : \mathbb{P}\, X$ through a relation $r : X \leftrightarrow Y$ is the set of values of type Y that are related under r to a value in a.

B.5.15 Overriding

function 50 leftassoc (_ ⊕ _)

$$
\begin{array}{l}
[X, Y] \\
\_ \oplus \_ : (X \leftrightarrow Y) \times (X \leftrightarrow Y) \to (X \leftrightarrow Y) \\
\hline
\forall\, r, s : X \leftrightarrow Y \bullet r \oplus s = ((\mathit{dom}\, s) \lhd r) \cup s
\end{array}
$$

If r and s are both relations between X and Y, the overriding of r by s is the whole of s together with those members of r that have no first components that are in the domain of s.

B.5.16 Transitive closure

function 90 (_ $^+$)

$$
\begin{array}{l}
[X] \\
\_\,^{+} : (X \leftrightarrow X) \to (X \leftrightarrow X) \\
\hline
\forall\, r : X \leftrightarrow X \bullet r\,^{+} = \bigcap \{\ s : X \leftrightarrow X \mid r \subseteq s \land r \mathbin{;} s \subseteq s\ \}
\end{array}
$$

The transitive closure of a relation $r : X \leftrightarrow X$ is the smallest set that contains r and is closed under the action of composing r with its members.

B.5.17 Reflexive transitive closure

function 90 (_ *)

$$
\begin{array}{l}
[X] \\
\_\,^{*} : (X \leftrightarrow X) \to (X \leftrightarrow X) \\
\hline
\forall\, r : X \leftrightarrow X \bullet r\,^{*} = r\,^{+} \cup \mathit{id}\, X
\end{array}
$$

The reflexive transitive closure of a relation $r : X \leftrightarrow X$ is the relation formed by extending the transitive closure of r by the identity relation on X.

B.6 Functions

section *function_toolkit* parents *relation_toolkit*

B.6.1 Partial functions

generic 5 rightassoc $(\_ \pfun \_)$

$$X \pfun Y == \{ f : X \leftrightarrow Y \mid \forall p, q : f \mid p.1 = q.1 \bullet p.2 = q.2 \}$$

$X \pfun Y$ is the set of all partial functions from X to Y, that is, the set of all relations between X and Y such that each x in X is related to at most one y in Y. The terms "function" and "partial function" are synonymous.

B.6.2 Partial injections

generic 5 rightassoc $(\_ \pinj \_)$

$$X \pinj Y == \{ f : X \leftrightarrow Y \mid \forall p, q : f \bullet p.1 = q.1 \Leftrightarrow p.2 = q.2 \}$$

$X \pinj Y$ is the set of partial injections from X to Y, that is, the set of all relations between X and Y such that each x in X is related to no more than one y in Y, and each y in Y is related to no more than one x in X. The terms "injection" and "partial injection" are synonymous.

B.6.3 Total injections

generic 5 rightassoc $(\_ \rightarrowtail \_)$

$$X \rightarrowtail Y == (X \pinj Y) \cap (X \rightarrow Y)$$

$X \rightarrowtail Y$ is the set of total injections from X to Y, that is, the set of injections from X to Y that are also total functions from X to Y.

B.6.4 Partial surjections

generic 5 rightassoc $(\_ \psurj \_)$

$$X \psurj Y == \{ f : X \pfun Y \mid \mathit{ran}\, f = Y \}$$

$X \psurj Y$ is the set of partial surjections from X to Y, that is, the set of functions from X to Y whose range is equal to Y. The terms "surjection" and "partial surjection" are synonymous.

B.6.5 Total surjections

generic 5 rightassoc $(\_ \twoheadrightarrow \_)$

$$X \twoheadrightarrow Y == (X \psurj Y) \cap (X \rightarrow Y)$$

$X \twoheadrightarrow Y$ is the set of total surjections from X to Y, that is, the set of surjections from X to Y that are also total functions from X to Y.

B.6.6 Bijections

generic 5 rightassoc $(\_ \bij \_)$

$$X \bij Y == (X \twoheadrightarrow Y) \cap (X \rightarrowtail Y)$$

$X \bij Y$ is the set of bijections from X to Y, that is, the set of total surjections from X to Y that are also total injections from X to Y.

B.6.7 Finite functions

generic 5 rightassoc $(\_ \mathrel{+\!\!\!\!\!\!\twoheadrightarrow} \_)$

$$X \mathrel{+\!\!\!\!\!\!\twoheadrightarrow} Y == (X \mathrel{+\!\!\!\!\!\rightarrow} Y) \cap \mathbb{F}(X \times Y)$$

The finite functions from X to Y are the functions from X to Y that are also finite sets.

B.6.8 Finite injections

generic 5 rightassoc $(\_ \rightarrowtail\!\!\!\!\!\!\!\twoheadrightarrow \_)$

$$X \rightarrowtail\!\!\!\!\!\!\!\twoheadrightarrow Y == (X \mathrel{+\!\!\!\!\!\!\twoheadrightarrow} Y) \cap (X \mathrel{\rightarrowtail\!\!\!\!\!\!+} Y)$$

The finite injections from X to Y are the injections from X to Y that are also finite functions from X to Y.

B.6.9 Disjointness

relation $(\mathit{disjoint}\ \_)$

$$
\begin{array}{|l}
[L, X] \\
\mathit{disjoint}\ \_ : \mathbb{P}(L \leftrightarrow \mathbb{P}\, X) \\
\hline
\forall f : L \leftrightarrow \mathbb{P}\, X \bullet \mathit{disjoint}\, f \Leftrightarrow (\forall p, q : f \mid p \neq q \bullet p.2 \cap q.2 = \varnothing)
\end{array}
$$

A labelled family of sets is disjoint precisely when any distinct pair yields sets with no members in common.

B.6.10 Partitions

relation $(\_\ \mathit{partition}\ \_)$

$$
\begin{array}{|l}
[L, X] \\
\_\ \mathit{partition}\ \_ : (L \leftrightarrow \mathbb{P}\, X) \leftrightarrow \mathbb{P}\, X \\
\hline
\forall f : L \leftrightarrow \mathbb{P}\, X;\ a : \mathbb{P}\, X \bullet f\ \mathit{partition}\ a \Leftrightarrow \mathit{disjoint}\, f \wedge \bigcup(\mathit{ran}\, f) = a
\end{array}
$$

A labelled family of sets f partitions a set a precisely when f is disjoint and the union of all the sets in f is a.

B.7 Numbers

section $\mathit{number\_toolkit}$

B.7.1 Successor

function 80 $(\mathit{succ}\ \_)$

$$
\begin{array}{|l}
\mathit{succ}\ \_ : \mathbb{P}(\mathbb{N} \times \mathbb{N}) \\
\hline
(\mathit{succ}\ \_) = \lambda\, n : \mathbb{N} \bullet n + 1
\end{array}
$$

The successor of a natural number n is equal to $n + 1$.

B.7.2 Integers

$$\mathbb{Z} : \mathbb{P}\,\mathbb{A}$$

$\mathbb{Z}$ is the set of integers, that is, positive and negative whole numbers and zero. The set $\mathbb{Z}$ is characterised by axioms for its additive structure given in the prelude (clause 11) together with the next formal paragraph below.

Number systems that extend the integers may be specified as supersets of $\mathbb{Z}$.

B.7.3 Addition of integers, arithmetic negation

function 80 (- _)

$$\begin{array}{l}
\text{-}\,_ : \mathbb{P}(\mathbb{A} \times \mathbb{A}) \\
\hline
\forall x, y : \mathbb{Z} \bullet \exists_1 z : \mathbb{Z} \bullet ((x, y), z) \in (_ + _) \\
\forall x : \mathbb{Z} \bullet \exists_1 y : \mathbb{Z} \bullet (x, y) \in (\text{-}\,_) \\
\forall i, j, k : \mathbb{Z} \bullet \\
\qquad (i + j) + k = i + (j + k) \\
\qquad \wedge\ i + j = j + i \\
\qquad \wedge\ i + \text{-}i = 0 \\
\qquad \wedge\ i + 0 = i \\
\mathbb{Z} = \{z : \mathbb{A} \mid \exists x : \mathbb{N} \bullet z = x \vee z = \text{-}x\}
\end{array}$$

The binary addition operator (_+_) is defined in the prelude (clause 11). The definition here introduces additional properties for integers. The addition and negation operations on integers are total functions that take integer values. The integers form a commutative group under (_ + _) with (- _) as the inverse operation and 0 as the identity element.

NOTE If *function_toolkit* notation were exploited, the negation operator could be defined as follows.

$$\begin{array}{l}
\text{-}\,_ : \mathbb{A} \leftrightarrow \mathbb{A} \\
\hline
(\mathbb{Z} \times \mathbb{Z}) \lhd (_ + _) \in \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \\
\mathbb{Z} \lhd (\text{-}\,_) \in \mathbb{Z} \rightarrow \mathbb{Z} \\
\forall i, j, k : \mathbb{Z} \bullet \\
\qquad (i + j) + k = i + (j + k) \\
\qquad \wedge\ i + j = j + i \\
\qquad \wedge\ i + \text{-}i = 0 \\
\qquad \wedge\ i + 0 = i \\
\forall h : \mathbb{P}\,\mathbb{Z} \bullet \\
\qquad 1 \in h \wedge (\forall i, j : h \bullet i + j \in h \wedge \text{-}i \in h) \\
\qquad\qquad \Rightarrow h = \mathbb{Z}
\end{array}$$

B.7.4 Subtraction

function 30 leftassoc (_ − _)

$$\begin{array}{l}
_ - _ : \mathbb{P}((\mathbb{A} \times \mathbb{A}) \times \mathbb{A}) \\
\hline
\forall x, y : \mathbb{Z} \bullet \exists_1 z : \mathbb{Z} \bullet ((x, y), z) \in (_ - _) \\
\forall i, j : \mathbb{Z} \bullet i - j = i + \text{-}j
\end{array}$$

Subtraction is a function whose domain includes all pairs of integers. For all integers i and j, $i - j$ is equal to $i + \text{-}j$.

NOTE If *function_toolkit* notation were exploited, the subtraction operator could be defined as follows.

$$\begin{array}{|l}
_ - _ : \mathbb{A} \times \mathbb{A} \nvrightarrow \mathbb{A} \\
\hline
(\mathbb{Z} \times \mathbb{Z}) \lhd (_ - _) \in \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \\
\forall\, i, j : \mathbb{Z} \bullet i - j = i + \text{-}j
\end{array}$$

B.7.5 Less-than-or-equal

relation $(\_ \leq \_)$

$$\begin{array}{|l}
_ \leq _ : \mathbb{P}(\mathbb{A} \times \mathbb{A}) \\
\hline
\forall\, i, j : \mathbb{Z} \bullet i \leq j \Leftrightarrow j - i \in \mathbb{N}
\end{array}$$

For all integers i and j, $i \leq j$ if and only if their difference $j - i$ is a natural number.

B.7.6 Less-than

relation $(\_ < \_)$

$$\begin{array}{|l}
_ < _ : \mathbb{P}(\mathbb{A} \times \mathbb{A}) \\
\hline
\forall\, i, j : \mathbb{Z} \bullet i < j \Leftrightarrow i + 1 \leq j
\end{array}$$

For all integers i and j, $i < j$ if and only if $i + 1 \leq j$.

B.7.7 Greater-than-or-equal

relation $(\_ \geq \_)$

$$\begin{array}{|l}
_ \geq _ : \mathbb{P}(\mathbb{A} \times \mathbb{A}) \\
\hline
\forall\, i, j : \mathbb{Z} \bullet i \geq j \Leftrightarrow j \leq i
\end{array}$$

For all integers i and j, $i \geq j$ if and only if $j \leq i$.

B.7.8 Greater-than

relation $(\_ > \_)$

$$\begin{array}{|l}
_ > _ : \mathbb{P}(\mathbb{A} \times \mathbb{A}) \\
\hline
\forall\, i, j : \mathbb{Z} \bullet i > j \Leftrightarrow j < i
\end{array}$$

For all integers i and j, $i > j$ if and only if $j < i$.

B.7.9 Strictly positive natural numbers

$$\mathbb{N}_1 == \{ x : \mathbb{N} \mid \neg\, x = 0 \}$$

The strictly positive natural numbers $\mathbb{N}_1$ are the natural numbers except zero.

B.7.10 Non-zero integers

$$\mathbb{Z}_1 == \{x : \mathbb{Z} \mid \neg\, x = 0\}$$

The non-zero integers $\mathbb{Z}_1$ are the integers except zero.

B.7.11 Multiplication of integers

function 40 leftassoc $(\_ * \_)$

$$
\begin{array}{l}
\_ * \_ : \mathbb{P}((\mathbb{A} \times \mathbb{A}) \times \mathbb{A}) \\
\hline
\forall\, x, y : \mathbb{Z} \bullet \exists_1\, z : \mathbb{Z} \bullet ((x, y), z) \in (\_ * \_) \\
\forall\, i, j, k : \mathbb{Z} \bullet \\
\qquad (i * j) * k = i * (j * k) \\
\qquad \land\ i * j = j * i \\
\qquad \land\ i * (j + k) = i * j + i * k \\
\qquad \land\ 0 * i = 0 \\
\qquad \land\ 1 * i = i
\end{array}
$$

The binary multiplication operator $(\_ * \_)$ is defined for integers. The multiplication operation on integers is a total function and has integer values. Multiplication on integers is characterised by the unique operation under which the integers become a commutative ring with identity element 1.

NOTE If *function_toolkit* notation were exploited, the multiplication operator could be defined as follows.

$$
\begin{array}{l}
\_ * \_ : (\mathbb{A} \times \mathbb{A}) \mathrel{+\!\!\!\rightarrow} \mathbb{A} \\
\hline
(\mathbb{Z} \times \mathbb{Z}) \lhd (\_ * \_) \in \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z} \\
\forall\, i, j, k : \mathbb{Z} \bullet \\
\qquad (i * j) * k = i * (j * k) \\
\qquad \land\ i * j = j * i \\
\qquad \land\ i * (j + k) = i * j + i * k \\
\qquad \land\ 0 * i = 0 \\
\qquad \land\ 1 * i = i
\end{array}
$$

B.7.12 Division, modulus

function 40 leftassoc $(\_\ \mathit{div}\ \_)$
function 40 leftassoc $(\_\ \mathit{mod}\ \_)$

$$
\begin{array}{l}
\_\ \mathit{div}\ \_, \_\ \mathit{mod}\ \_ : \mathbb{P}((\mathbb{A} \times \mathbb{A}) \times \mathbb{A}) \\
\hline
\forall\, x : \mathbb{Z};\ y : \mathbb{Z}_1 \bullet \exists_1\, z : \mathbb{Z} \bullet ((x, y), z) \in (\_\ \mathit{div}\ \_) \\
\forall\, x : \mathbb{Z};\ y : \mathbb{Z}_1 \bullet \exists_1\, z : \mathbb{Z} \bullet ((x, y), z) \in (\_\ \mathit{mod}\ \_) \\
\forall\, i : \mathbb{Z};\ j : \mathbb{Z}_1 \bullet \\
\qquad i = (i\ \mathit{div}\ j) * j + i\ \mathit{mod}\ j \\
\qquad \land\ (0 \leq i\ \mathit{mod}\ j < j \lor j < i\ \mathit{mod}\ j \leq 0)
\end{array}
$$

For all integers i and non-zero integers j, the pair (i, j) is in the domain of $\_\mathit{div}\_$ and of $\_\mathit{mod}\_$, and $i\ \mathit{div}\ j$ and $i\ \mathit{mod}\ j$ have integer values.

When not zero, $i\ \mathit{mod}\ j$ has the same sign as j. This means that $i\ \mathit{div}\ j$ is the largest integer no greater than the rational number i/j.

NOTE If *function_toolkit* notation were exploited, the division and modulus operators could be defined as follows.

$$
\begin{array}{|l}
\_\, div\, \_, \_\, mod\, \_ : \mathbb{A} \times \mathbb{A} \leftrightarrow \mathbb{A} \\
\hline
(\mathbb{Z} \times \mathbb{Z}_1) \lhd (\_\, div\, \_) \in \mathbb{Z} \times \mathbb{Z}_1 \rightarrow \mathbb{Z} \\
(\mathbb{Z} \times \mathbb{Z}_1) \lhd (\_\, mod\, \_) \in \mathbb{Z} \times \mathbb{Z}_1 \rightarrow \mathbb{Z} \\
\forall i : \mathbb{Z};\ j : \mathbb{Z}_1 \bullet \\
\qquad i = (i\ div\ j) * j + i\ mod\ j \\
\qquad \land (0 \leq i\ mod\ j < j \lor j < i\ mod\ j \leq 0)
\end{array}
$$

B.8 Sequences

section *sequence_toolkit* parents *function_toolkit*, *number_toolkit*

B.8.1 Number range

function 20 leftassoc (_ .. _)

$$
\begin{array}{|l}
\_ \mathrel{..} \_ : \mathbb{A} \times \mathbb{A} \leftrightarrow \mathbb{P}\,\mathbb{A} \\
\hline
(\mathbb{Z} \times \mathbb{Z}) \lhd (\_ \mathrel{..} \_) \in \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{P}\,\mathbb{Z} \\
\forall i, j : \mathbb{Z} \bullet i \mathrel{..} j = \{\ k : \mathbb{Z} \mid i \leq k \leq j\ \}
\end{array}
$$

The number range from i to j is the set of all integers greater than or equal to i, which are also less than or equal to j.

B.8.2 Iteration

$$
\begin{array}{|l}
[X] \\
iter : \mathbb{Z} \rightarrow (X \leftrightarrow X) \rightarrow (X \leftrightarrow X) \\
\hline
\forall r : X \leftrightarrow X \bullet iter\ 0\ r = id\ X \\
\forall r : X \leftrightarrow X;\ n : \mathbb{N} \bullet iter\ (n+1)\ r = r \circ (iter\ n\ r) \\
\forall r : X \leftrightarrow X;\ n : \mathbb{N} \bullet iter\ (\text{-}n)\ r = iter\ n\ (r^{\sim})
\end{array}
$$

iter is the iteration function for a relation. The iteration of a relation $r : X \leftrightarrow X$ for zero times is the identity relation on X. The iteration of a relation $r : X \leftrightarrow X$ for $n+1$ times is the composition of the relation with its iteration n times. The iteration of a relation $r : X \leftrightarrow X$ for -n times is the iteration for n times of the inverse of the relation.

function 90 (_ $^{-}$)

$$
\begin{array}{|l}
[X] \\
\_^{\,-} : (X \leftrightarrow X) \times \mathbb{Z} \rightarrow (X \leftrightarrow X) \\
\hline
\forall r : X \leftrightarrow X;\ n : \mathbb{N} \bullet r^n = iter\ n\ r
\end{array}
$$

$iter\ n\ r$ may be written as r^n.

B.8.3 Number of members of a set

function 80 ($\#\ \_$)

$$
\begin{array}{l}
[X] \\
\hline
\#\ \_ : \mathbb{F}\, X \rightarrow \mathbb{N} \\
\hline
\forall\, a : \mathbb{F}\, X \bullet \# a = (\mu\, n : \mathbb{N} \mid (\exists\, f : 1 \mathinner{.\,.} n \rightarrowtail a \bullet \mathit{ran}\, f = a))
\end{array}
$$

The number of members of a finite set is the upper limit of the number range starting at 1 that can be put into bijection with the set.

B.8.4 Minimum

function 80 ($\mathit{min}\ \_$)

$$
\begin{array}{l}
\mathit{min}\ \_ : \mathbb{P}\, \mathbb{A} \nrightarrow \mathbb{A} \\
\hline
\mathbb{P}\, \mathbb{Z} \lhd (\mathit{min}\ \_) = \{\ a : \mathbb{P}\, \mathbb{Z};\ m : \mathbb{Z} \mid m \in a \wedge (\forall\, n : a \bullet m \leq n) \bullet a \mapsto m\ \}
\end{array}
$$

If a set of integers has a member that is less than or equal to all members of that set, that member is its minimum.

B.8.5 Maximum

function 80 ($\mathit{max}\ \_$)

$$
\begin{array}{l}
\mathit{max}\ \_ : \mathbb{P}\, \mathbb{A} \nrightarrow \mathbb{A} \\
\hline
\mathbb{P}\, \mathbb{Z} \lhd (\mathit{max}\ \_) = \{\ a : \mathbb{P}\, \mathbb{Z};\ m : \mathbb{Z} \mid m \in a \wedge (\forall\, n : a \bullet n \leq m) \bullet a \mapsto m\ \}
\end{array}
$$

If a set of integers has a member that is greater than or equal to all members of that set, that member is its maximum.

B.8.6 Finite sequences

generic 80 ($\mathit{seq}\ \_$)

$$\mathit{seq}\, X == \{\ f : \mathbb{N} \mathrel{+\!\!\!\!\nrightarrow} X \mid \mathit{dom}\, f = 1 \mathinner{.\,.} \# f\ \}$$

A finite sequence is a finite indexed set of values of the same type, whose domain is a contiguous set of positive integers starting at 1.

$\mathit{seq}\, X$ is the set of all finite sequences of values of X, that is, of finite functions from the set $1 \mathinner{.\,.} n$, for some n, to elements of X.

B.8.7 Non-empty finite sequences

$$\mathit{seq}_1\, X == \mathit{seq}\, X \setminus \{\varnothing\}$$

$\mathit{seq}_1\, X$ is the set of all non-empty finite sequences of values of X.

B.8.8 Injective sequences

generic 80 ($\mathit{iseq}\ \_$)

$$\mathit{iseq}\, X == \mathit{seq}\, X \cap (\mathbb{N} \rightarrowtail\!\!\!\!\nrightarrow X)$$

$\mathit{iseq}\, X$ is the set of all injective finite sequences of values of X, that is, of finite sequences over X that are also injections.

B.8.9 Sequence brackets

function (⟨ ,, ⟩)

$$\langle \_ \rangle [X] == \lambda\, s : seq\, X \bullet s$$

The brackets ⟨ and ⟩ can be used for enumerated sequences.

B.8.10 Concatenation

function 30 leftassoc (_ ⁀ _)

$$[X]$$
$$\_ \frown \_ : seq\, X \times seq\, X \rightarrow seq\, X$$
$$\forall\, s, t : seq\, X \bullet s \frown t = s \cup \{\ n : dom\, t \bullet n + \#s \mapsto t\ n\ \}$$

Concatenation is a function of a pair of finite sequences of values of the same type whose result is a sequence that begins with all elements of the first sequence and continues with all elements of the second sequence.

B.8.11 Reverse

$$[X]$$
$$rev : seq\, X \rightarrow seq\, X$$
$$\forall\, s : seq\, X \bullet rev\ s = \lambda\, n : dom\, s \bullet s(\#s - n + 1)$$

The reverse of a sequence is the sequence obtained by taking its elements in the opposite order.

B.8.12 Head of a sequence

$$[X]$$
$$head : seq_1\, X \rightarrow X$$
$$\forall\, s : seq_1\, X \bullet head\ s = s\ 1$$

If s is a non-empty sequence of values, then $head\ s$ is the value that is first in the sequence.

B.8.13 Last of a sequence

$$[X]$$
$$last : seq_1\, X \rightarrow X$$
$$\forall\, s : seq_1\, X \bullet last\ s = s(\#s)$$

If s is a non-empty sequence of values, then $last\ s$ is the value that is last in the sequence.

B.8.14 Tail of a sequence

$$[X]$$
$$tail : seq_1\, X \rightarrow seq\, X$$
$$\forall\, s : seq_1\, X \bullet tail\ s = \lambda\, n : 1 \mathrel{..} (\#s - 1) \bullet s(n + 1)$$

If s is a non-empty sequence of values, then $tail\ s$ is the sequence of values that is obtained from s by discarding the first element and renumbering the remainder.

B.8.15 Front of a sequence

$$
\begin{array}{l}
[X] \\
\hline
front : seq_1\, X \rightarrow seq\, X \\
\hline
\forall s : seq_1\, X \bullet front\ s = \{\#s\} \ntriangleleft s
\end{array}
$$

If s is a non-empty sequence of values, then $front\ s$ is the sequence of values that is obtained from s by discarding the last element.

B.8.16 Squashing

$$
\begin{array}{l}
[X] \\
\hline
squash : (\mathbb{Z} \twoheadrightarrow\!\!\!\!\!\!\shortmid\;\; X) \rightarrow seq\, X \\
\hline
\forall f : \mathbb{Z} \twoheadrightarrow\!\!\!\!\!\!\shortmid\;\; X \bullet squash\ f = \{\ p : f \bullet \#\{\ i : dom\, f \mid i \leq p.1\ \} \mapsto p.2\ \}
\end{array}
$$

$squash$ takes a finite function $f : \mathbb{Z} \twoheadrightarrow\!\!\!\!\!\!\shortmid\;\; X$ and renumbers its domain to produce a finite sequence.

B.8.17 Extraction

function 45 rightassoc ($\_ \upharpoonleft \_$)

$$
\begin{array}{l}
[X] \\
\hline
\_ \upharpoonleft \_ : \mathbb{P}\,\mathbb{Z} \times seq\, X \rightarrow seq\, X \\
\hline
\forall a : \mathbb{P}\,\mathbb{Z};\ s : seq\, X \bullet a \upharpoonleft s = squash(a \lhd s)
\end{array}
$$

The extraction of a set a of indices from a sequence is the sequence obtained from the original by discarding any indices that are not in the set a, then renumbering the remainder.

B.8.18 Filtering

function 40 leftassoc ($\_ \upharpoonright \_$)

$$
\begin{array}{l}
[X] \\
\hline
\_ \upharpoonright \_ : seq\, X \times \mathbb{P}\, X \rightarrow seq\, X \\
\hline
\forall s : seq\, X;\ a : \mathbb{P}\, X \bullet s \upharpoonright a = squash(s \rhd a)
\end{array}
$$

The filter of a sequence by a set a is the sequence obtained from the original by discarding any members that are not in the set a, then renumbering the remainder.

B.8.19 Prefix relation

relation ($\_\ prefix\ \_$)

$$
\begin{array}{l}
[X] \\
\hline
\_\ prefix\ \_ : seq\, X \leftrightarrow seq\, X \\
\hline
\forall s, t : seq\, X \bullet s\ prefix\ t \Leftrightarrow s \subseteq t
\end{array}
$$

A sequence s is a prefix of another sequence t if it forms the front portion of t.

B.8.20 Suffix relation

relation (_ *suffix* _)

$$
\begin{array}{l}
[X] \\
\_\ suffix\ \_ : seq\ X \leftrightarrow seq\ X \\
\hline
\forall s, t : seq\ X \bullet s\ suffix\ t \Leftrightarrow (\exists u : seq\ X \bullet u \frown s = t)
\end{array}
$$

A sequence s is a suffix of another sequence t if it forms the end portion of t.

B.8.21 Infix relation

relation (_ *infix* _)

$$
\begin{array}{l}
[X] \\
\_\ infix\ \_ : seq\ X \leftrightarrow seq\ X \\
\hline
\forall s, t : seq\ X \bullet s\ infix\ t \Leftrightarrow (\exists u, v : seq\ X \bullet u \frown s \frown v = t)
\end{array}
$$

A sequence s is an infix of another sequence t if it forms a mid portion of t.

B.8.22 Distributed concatenation

$$
\begin{array}{l}
[X] \\
\frown/ : seq\ seq\ X \rightarrow seq\ X \\
\hline
\frown/\ \langle\ \rangle = \langle\ \rangle \\
\forall s : seq\ X \bullet \frown/\ \langle s \rangle = s \\
\forall q, r : seq\ seq\ X \bullet \frown/(q \frown r) = (\frown/\ q) \frown (\frown/\ r)
\end{array}
$$

The distributed concatenation of a sequence t of sequences of values of type X is the sequence of values of type X that is obtained by concatenating the members of t in order.

B.9 Standard toolkit

section *standard_toolkit* parents *sequence_toolkit*

The standard toolkit contains the definitions of section *sequence_toolkit* (and implicitly those of its ancestral sections).

Annex C
(normative)

Organisation by concrete syntax production

C.1 Introduction

This annex duplicates some of the definitions presented in the normative clauses, but re-organised by concrete syntax production. This re-organisation provides no suitable place to accommodate the material listed in the rest of this introduction. That material is consequently omitted from this annex.

a) From Concrete syntax, the rules defining:

   1) `Formals`, used in Generic axiomatic description paragraph, Generic schema paragraph, Generic horizontal definition paragraph, and Generic conjecture paragraph;

   2) `DeclName`, used in `Branch`, Schema hiding expression, Schema renaming expression, Colon declaration and Equal declaration;

   3) `RefName`, used in Reference expression, Generic instantiation expression, and Binding selection expression;

   4) `OpName` and its auxiliaries, used in `RefName` and `DeclName`;

   5) `ExpSep` and `ExpressionList`, used in auxiliaries of Relation operator application predicates and Function and generic operator application expressions;

   6) and also the operator precedences and associativities and additional syntactic restrictions.

b) From Characterisation rules:

   1) Characteristic tuple.

c) From Prelude:

   1) its text is relevant not just to number literal expressions but also to the list arguments in Relation operator application predicates and Function and generic operator application expressions.

d) From Syntactic transformation rules:

   1) `Name` and `ExpressionList`.

e) From Type inference rules:

   1) Carrier set and Generic type instantiation.

   2) Summary of scope rules.

f) From Semantic relations:

   1) all of the relations for `Type` are omitted.

Also, the description of the overall effect of a phase, or how the phase operates, is generally omitted from this annex.

Moreover, some of the phases and representations are entirely omitted here, namely Mark-ups, Z characters, Lexis and Annotated syntax.

C.2 Specification

C.2.1 Introduction

`Specification` is the start symbol of the syntax. A `Specification` can be either a sectioned specification or an anonymous specification. A sectioned specification comprises a sequence of named sections. An anonymous specification comprises a single anonymous section.

C.2.2 Sectioned specification

C.2.2.1 Syntax

```
Specification     = { Section }
                  | ...
                  ;
```

C.2.2.2 Type

$$\frac{\{\} \vdash^{\mathcal{S}} s_{prelude} \mathbin{⦂} \Gamma_0 \qquad \delta_1 \vdash^{\mathcal{S}} s_1 \mathbin{⦂} \Gamma_1 \qquad \ldots \qquad \delta_n \vdash^{\mathcal{S}} s_n \mathbin{⦂} \Gamma_n}{\vdash^{\mathcal{Z}} s_1 \mathbin{⦂} \Gamma_1 \ldots s_n \mathbin{⦂} \Gamma_n} \left(\begin{array}{l} \delta_1 = \{prelude \mapsto \Gamma_0\} \\ \vdots \\ \delta_n = \delta_{n-1} \cup \{i_{n-1} \mapsto \Gamma_{n-1}\} \end{array} \right)$$

where i_{n-1} is the name of section s_{n-1}, and none of the sections $s_1 \ldots s_n$ are named *prelude*.

Each section is typechecked in an environment formed from preceding sections, and is annotated with an environment that it establishes.

NOTE The environment established by the prelude section is as follows.

$$\begin{array}{rcl} \Gamma_0 & = & (\mathbb{A}, (prelude, \mathbb{P}(\mathtt{GIVEN}\ \mathbb{A}))), \\ & & (\mathbb{N}, (prelude, \mathbb{P}(\mathtt{GIVEN}\ \mathbb{A}))), \\ & & (number\_literal\_0, (prelude, (\mathtt{GIVEN}\ \mathbb{A}))), \\ & & (number\_literal\_1, (prelude, (\mathtt{GIVEN}\ \mathbb{A}))), \\ & & (\bowtie{+}\bowtie, (prelude, \mathbb{P}(((\mathtt{GIVEN}\ \mathbb{A}) \times (\mathtt{GIVEN}\ \mathbb{A})) \times (\mathtt{GIVEN}\ \mathbb{A})))) \end{array}$$

If one of the sections $s_1 \ldots s_n$ is named *prelude*, then the same type inference rule applies except that the type subsequent for the prelude section is omitted.

C.2.2.3 Semantics

$$[\![\ s_1 \ldots s_n\ ]\!]^{\mathcal{Z}} \quad = \quad ([\![\ \text{section}\ \ prelude\ldots\ ]\!]^{\mathcal{S}} \mathbin{⨾} [\![\ s_1\ ]\!]^{\mathcal{S}} \mathbin{⨾} \ldots \mathbin{⨾} [\![\ s_n\ ]\!]^{\mathcal{S}})\ \varnothing$$

The meaning of the Z specification $s_1 \ldots s_n$ is the function from sections' names to their sets of models formed by starting with the empty function and extending that with a maplet from a section's name to its set of models for each section in the specification, starting with the prelude.

To determine $[\![\ \text{section}\ \ prelude\ldots\ ]\!]^{\mathcal{Z}}$ another prelude shall not be prefixed onto it.

NOTE The meaning of a specification is not the meaning of its last section, so as to permit several meaningful units within a single document.

C.2.3 Anonymous specification

C.2.3.1 Syntax

```
Specification     = ...
                  | { Paragraph }
                  ;
```

C.2.3.2 Transformation

The anonymous specification d_1 ... d_n is semantically equivalent to the sectioned specification comprising a single section containing those paragraphs with the mathematical toolkit of annex B as its parent.

$$d_1 \ldots d_n \implies \textit{Mathematical toolkit} \text{ section } \textit{Specification} \text{ parents } \textit{standard\_toolkit} \texttt{ END } d_1 \ldots d_n$$

In this transformation, *Mathematical toolkit* denotes the entire text of annex B. The name given to the section is not important: it need not be *Specification*, though it may not be *prelude* or that of any section of the mathematical toolkit.

C.3 Section

C.3.1 Introduction

A `Section` can be either an inheriting section or a base section. An inheriting section gathers together the paragraphs of parent sections with new paragraphs. A base section is like an inheriting section but has no parents.

C.3.2 Inheriting section

C.3.2.1 Syntax

```
Section           = section , NAME , parents , [ NAME , { ,-tok , NAME } ] , END ,
                        { Paragraph }
                  | ...
                  ;
```

C.3.2.2 Type

$$\frac{\beta_0 \vdash^{\mathcal{D}} d_1 \mathbin{⨾} \sigma_1 \quad \ldots \quad \beta_{n-1} \vdash^{\mathcal{D}} d_n \mathbin{⨾} \sigma_n}{\Lambda \vdash^{\mathcal{S}} \text{section } i \text{ parents } i_1, \ldots, i_m \texttt{ END } d_1 \mathbin{⨾} \sigma_1 \ldots d_n \mathbin{⨾} \sigma_n \mathbin{⨾} \Gamma} \left(\begin{array}{l} i \notin \mathit{dom}\ \Lambda \\ \{i_1, \ldots, i_m\} \subseteq \mathit{dom}\ \Lambda \\ \gamma_{-1} = \text{if } i = \mathit{prelude} \text{ then } \{\} \text{ else } \Lambda\ \mathit{prelude} \\ \gamma_0 = \gamma_{-1} \cup \Lambda\ i_1 \cup \ldots \cup \Lambda\ i_m \\ \beta_0 = \gamma_0 \mathbin{⨾} \mathit{second} \\ \mathit{disjoint}\ \langle \mathit{dom}\ \sigma_1, \ldots, \mathit{dom}\ \sigma_n \rangle \\ \Gamma \in (\_ \mathbin{⇸} \_) \\ \Gamma = \gamma_0 \cup \{j : \texttt{NAME};\ \tau : \texttt{Type} \mid j \mapsto \tau \in \sigma_1 \cup \ldots \cup \sigma_n \bullet j \mapsto (i, \tau)\} \\ \beta_1 = \beta_0 \cup \sigma_1 \\ \quad \vdots \\ \beta_{n-1} = \beta_{n-2} \cup \sigma_{n-1} \end{array} \right)$$

Taking the side-conditions in order, this type inference rule ensures that:

a) the name of the section, i, is different from that of any previous section;

b) the names in the parents list are names of known sections;

c) the section environment of the prelude is included if the section is not itself the prelude;

d) the section environment γ_0 is formed from those of the parents;

e) the type environment β_0 is determined from the section environment γ_0;

f) there is no global redefinition between any pair of paragraphs of the section (the sets of names in their signatures are disjoint);

g) a name which is common to the environments of multiple parents shall have originated in a common ancestral section, and a name introduced by a paragraph of this section shall not also be introduced by another paragraph or parent section (all ensured by the combined environment being a function);

h) the annotation of the section is an environment formed from those of its parents extended according to the signatures of its paragraphs;

i) and the type environment in which a paragraph is typechecked is formed from that of the parent sections extended with the signatures of the preceding paragraphs of this section.

NOTE 1 Ancestors need not be immediate parents, and a section cannot be amongst its own ancestors (no cycles in the parent relation).

NOTE 2 The name of a section can be the same as the name of a variable introduced in a declaration—the two are not confused.

C.3.2.3 Semantics

The prelude section, as defined in clause 11, is treated specially, as it is the only one that does not have prelude as an implicit parent.

$$[\![\ \text{section}\ \ \mathit{prelude}\ \ \text{parents}\ \mathtt{END}\ d_1\ \ldots\ d_n\ ]\!]^{\mathcal{S}}$$
$$=$$
$$\lambda\ T : \mathit{SectionModels} \bullet \{\mathit{prelude} \mapsto ([\![\ d_1\ ]\!]^{\mathcal{D}} ⨾ \ldots ⨾ [\![\ d_n\ ]\!]^{\mathcal{D}})\ (\!|\ \{\varnothing\}\ |\!)\}$$

The meaning of the prelude section is given by that constant function which, whatever function from sections' names and their sets of models it is given, returns the singleton set mapping the name *prelude* to its set of models. The set of models is that to which the set containing an empty model is related by the composition of the relations between models that denote the meanings of each of the prelude's paragraphs—see clause 11 for details of those paragraphs.

NOTE One model of the prelude section can be written as follows.

$$\begin{array}{l}\{\mathbb{A} \mapsto \mathbb{A},\\ \mathbb{N} \mapsto \mathbb{N},\\ \mathit{number\_literal\_0} \mapsto 0,\\ \mathit{number\_literal\_1} \mapsto 1,\\ \_ + \_ \mapsto \{((0,0),0),((0,1),1),((1,0),1),((1,1),2),\ldots\}\}\end{array}$$

The behaviour of $(\_ + \_)$ on non-natural numbers, e.g. reals, has not been defined at this point, so the set of models for the prelude section includes alternatives for every possible extended behaviour of addition.

$$[\![\ \text{section}\ \ i\ \ \text{parents}\ \ i_1, \ldots, i_m\ \mathtt{END}\ d_1\ \ldots\ d_n\ ]\!]^{\mathcal{S}}$$
$$=$$
$$\lambda\ T : \mathit{SectionModels} \bullet T \cup \{i \mapsto$$
$$([\![\ d_1\ ]\!]^{\mathcal{D}} ⨾ \ldots ⨾ [\![\ d_n\ ]\!]^{\mathcal{D}})\ (\!|\ \{M_0 : T\ \mathit{prelude};\ M_1 : T\ i_1;\ \ldots;\ M_m : T\ i_m;\ M : \mathit{Model} \mid M = M_0 \cup M_1 \cup \ldots \cup M_m \bullet M\}\ |\!)\}$$

The meaning of a section other than the prelude is the extension of a function from sections' names to their sets of models with a maplet from the given section's name to its set of models. The given section's set of models is that to which the union of the models of the section's parents is related by the composition of the relations between models that denote the meanings of each of the section's paragraphs.

C.3.3 Base section

C.3.3.1 Syntax

```
Section          = ...
                 | section , NAME , END , { Paragraph }
                 ;
```

C.3.3.2 Transformation

The base section section i END d_1 ... d_n is semantically equivalent to the inheriting section that inherits from no parents (bar *prelude*).

$$\text{section } i \ \mathtt{END}\ d_1 \ ...\ d_n \quad \Longrightarrow \quad \text{section } i \ \text{parents}\ \mathtt{END}\ d_1 \ ...\ d_n$$

C.4 Paragraph

C.4.1 Introduction

A Paragraph can introduce new names into the models, and can constrain the values associated with names. A Paragraph can be any of given types, axiomatic description, schema definition, generic axiomatic description, generic schema definition, horizontal definition, generic horizontal definition, generic operator definition, free types, conjecture, generic conjecture, or operator template.

C.4.2 Given types

C.4.2.1 Syntax

```
Paragraph        = [-tok , NAME , { ,-tok , NAME } , ]-tok , END
                 | ...
                 ;
```

C.4.2.2 Type

$$\frac{}{\Sigma \vdash^{\mathcal{D}} [i_1, ..., i_n]\ \mathtt{END} \ ⨾\ \sigma} \left(\begin{array}{l} \#\{i_1,\ ...,\ i_n\} = n \\ \sigma = i_1 : \mathbb{P}(\mathtt{GIVEN}\ i_1);\ ...;\ i_n : \mathbb{P}(\mathtt{GIVEN}\ i_n) \end{array} \right)$$

In a given types paragraph, there shall be no duplication of names. The annotation of the paragraph is a signature associating the given type names with powerset types.

C.4.2.3 Semantics

The given types paragraph $[i_1, ..., i_n]$ END introduces unconstrained global names.

$$\begin{array}{rcl} [\![\ [i_1, ..., i_n]\ \mathtt{END}\ ]\!]^{\mathcal{D}} & = & \{M : \mathit{Model};\ w_1,\ ...,\ w_n : \mathbb{W} \\ & & \quad \bullet M \mapsto M \cup \{i_1 \mapsto w_1,\ ...,\ i_n \mapsto w_n\} \\ & & \qquad \cup \{i_1\ \mathit{decor}\ \heartsuit \mapsto w_1,\ ...,\ i_n\ \mathit{decor}\ \heartsuit \mapsto w_n\}\} \end{array}$$

It relates a model M to that model extended with associations between the names of the given types and semantic values chosen to represent their carrier sets. Associations for names decorated with the reserved stroke ♡ are also introduced, so that references to them from given types (15.2.6.1) can avoid being captured.

C.4.3 Axiomatic description

C.4.3.1 Syntax

```
Paragraph        = ...
                 | AX , SchemaText , END
                 | ...
                 ;
```

C.4.3.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathbin{⨾} \tau}{\Sigma \vdash^{\mathcal{D}} \texttt{AX}\ e \mathbin{⨾} \tau\ \texttt{END} \mathbin{⨾} \sigma}(\tau = \mathbb{P}[\sigma])$$

In an axiomatic description paragraph AX e END, the expression e shall be a schema. The annotation of the paragraph is the signature of that schema.

C.4.3.3 Semantics

The axiomatic description paragraph AX e END introduces global names and constraints on their values.

$$[\![\ \texttt{AX}\ e\ \texttt{END}\ ]\!]^{\mathcal{D}} \ = \ \{M : \mathit{Model};\ t : \mathbb{W} \mid t \in [\![\ e\ ]\!]^{\mathcal{E}} M \bullet M \mapsto M \cup t\}$$

It relates a model M to that model extended with a binding t of the schema that is the value of e in model M.

C.4.4 Schema definition

C.4.4.1 Syntax

```
Paragraph        = ...
                 | SCH , NAME , SchemaText , END
                 | ...
                 ;
```

C.4.4.2 Transformation

The schema definition paragraph SCH i t END introduces the global name i, associating it with the schema that is the value of t.

$$\texttt{SCH}\ i\ t\ \texttt{END} \quad\Longrightarrow\quad \texttt{AX}\ [i == t]\ \texttt{END}$$

The paragraph is semantically equivalent to the axiomatic description paragraph whose sole declaration associates the schema's name with the expression resulting from syntactic transformation of the schema text.

C.4.5 Generic axiomatic description

C.4.5.1 Syntax

```
Paragraph       = ...
                | GENAX , [-tok , Formals , ]-tok , SchemaText , END
                | ...
                ;
```

C.4.5.2 Type

$$\frac{\Sigma \oplus \{i_1 \mapsto \mathbb{P}(\mathtt{GENTYPE}\ i_1),\ ...,\ i_n \mapsto \mathbb{P}(\mathtt{GENTYPE}\ i_n)\} \vdash^{\mathcal{E}}\ e\ \raise0.5ex\hbox{\rm o}\kern-0.4em\lower0.2ex\hbox{\rm o}\ \tau}{\Sigma\ \vdash^{\mathcal{D}}\ \mathtt{GENAX}\ [i_1, ..., i_n]\ e\ ⨾\ \tau\ \mathtt{END}\ ⨾\ \sigma} \left(\begin{array}{l} \#\,\{i_1,\ ...,\ i_n\} = n \\ \tau = \mathbb{P}[\beta] \\ \sigma = \lambda\ j : dom\ \beta \bullet [i_1, ..., i_n]\ (\beta\ j) \end{array}\right)$$

In a generic axiomatic description paragraph GENAX $[i_1, ..., i_n]$ e END, there shall be no duplication of names within the generic parameters. The expression e is typechecked, in an environment overridden by the generic parameters, and shall be a schema. The annotation of the paragraph is formed from the signature of that schema, having the same names but associated with types that are generic.

C.4.5.3 Semantics

The generic axiomatic description paragraph GENAX $[i_1, ..., i_n]$ e END introduces global names and constraints on their values, with generic parameters that have to be instantiated (by sets) whenever those names are referenced.

$$\begin{array}{l}
[\![\ \mathtt{GENAX}\ [i_1, ..., i_n]\ (e\ ⨾\ \mathbb{P}[j_1 : \tau_1;\ ...;\ j_m : \tau_m])\ \mathtt{END}\ ]\!]^{\mathcal{D}} = \\
\qquad \{M : Model;\ u : \mathbb{W} \uparrow n \rightarrow \mathbb{W} \\
\qquad\quad \mid \forall\ w_1,\ ...,\ w_n : \mathbb{W} \bullet \exists\ w : \mathbb{W} \bullet \\
\qquad\qquad u\ (w_1,\ ...,\ w_n) \in w \\
\qquad\qquad \land (M \oplus \{i_1 \mapsto w_1,\ ...,\ i_n \mapsto w_n\} \cup \{i_1\ decor\ \spadesuit \mapsto w_1,\ ...,\ i_n\ decor\ \spadesuit \mapsto w_n\}) \mapsto w \in [\![\ e\ ]\!]^{\mathcal{E}} \\
\qquad\quad \bullet M \mapsto M \cup \lambda\ y : \{j_1,\ ...,\ j_m\} \bullet \lambda\ x : \mathbb{W} \uparrow n \bullet u\ x\ y\}
\end{array}$$

Given a model M and generic argument sets w_1, ..., w_n, the semantic value of the schema e in that model overridden by the association of the generic parameter names with those sets is w. All combinations of generic argument sets are considered. The function u maps the generic argument sets to a binding in the schema w. The paragraph relates the model M to that model extended with the binding that associates the names of the schema e (namely j_1, ..., j_m) with the corresponding value in the binding resulting from application of u to arbitrary instantiating sets x. Associations for names decorated with the reserved stroke ♠ are also introduced whilst determining the semantic value of e, so that references to them from generic types (15.2.6.2) can avoid being captured.

C.4.6 Generic schema definition

C.4.6.1 Syntax

```
Paragraph       = ...
                | GENSCH , NAME , [-tok , Formals , ]-tok , SchemaText , END
                | ...
                ;
```


C.4.6.2 Transformation

The generic schema definition paragraph GENSCH i $[i_1, ..., i_n]$ t END can be instantiated to produce a schema definition paragraph.

$$\texttt{GENSCH}\ i\ [i_1, ..., i_n]\ t\ \texttt{END} \quad \Longrightarrow \quad \texttt{GENAX}\ [i_1, ..., i_n]\ [i == t]\ \texttt{END}$$

It is semantically equivalent to the generic axiomatic description paragraph with the same generic parameters and whose sole declaration associates the schema's name with the expression resulting from syntactic transformation of the schema text.

C.4.7 Horizontal definition

C.4.7.1 Syntax

```
Paragraph        = ...
                 | DeclName , == , Expression , END
                 | ...
                 ;
```

C.4.7.2 Transformation

The horizontal definition paragraph $i == e$ END introduces the global name i, associating with it the value of e.

$$i == e\ \texttt{END} \quad \Longrightarrow \quad \texttt{AX}\ [i == e]\ \texttt{END}$$

It is semantically equivalent to the axiomatic description paragraph that introduces the same single declaration.

C.4.8 Generic horizontal definition

C.4.8.1 Syntax

```
Paragraph        = ...
                 | NAME , [-tok , Formals , ]-tok , == , Expression , END
                 | ...
                 ;
```

C.4.8.2 Transformation

The generic horizontal definition paragraph i $[i_1, ..., i_n] == e$ END can be instantiated to produce a horizontal definition paragraph.

$$i\ [i_1, ..., i_n] == e\ \texttt{END} \quad \Longrightarrow \quad \texttt{GENAX}\ [i_1, ..., i_n]\ [i == e]\ \texttt{END}$$

It is semantically equivalent to the generic axiomatic description paragraph with the same generic parameters and that introduces the same single declaration.

C.4.9 Generic operator definition

C.4.9.1 Syntax

```
Paragraph        = ...
                 | GenName , == , Expression , END
                 | ...
                 ;

GenName          = PrefixGenName
                 | PostfixGenName
                 | InfixGenName
                 | NofixGenName
                 ;

PrefixGenName    = PRE , NAME
                 | L , { NAME , ( ES | SS ) } , NAME , ( ERE | SRE ) , NAME
                 ;

PostfixGenName   = NAME , POST
                 | NAME , EL , { NAME , ( ES | SS ) } , NAME , ( ER | SR )
                 ;

InfixGenName     = NAME , I , NAME
                 | NAME , EL , { NAME , ( ES | SS ) } , NAME , ( ERE | SRE ) , NAME
                 ;

NofixGenName     = L , { NAME , ( ES | SS ) } , NAME , ( ER | SR ) ;
```

C.4.9.2 Transformation

All generic names are transformed to juxtapositions of `NAME`s and generic parameter lists. This causes the generic operator definition paragraphs in which they appear to become generic horizontal definition paragraphs, and thus be amenable to further syntactic transformation.

Each resulting `NAME` should be one for which there is an operator template paragraph in scope (see 12.2.8).

C.4.9.3 `PrefixGenName`

$$\begin{array}{rcl} pre\ i & \Longrightarrow & pre\bowtie\ [i] \\ ln\ i_1\ ess_1\ ...\ i_{n-2}\ ess_{n-2}\ i_{n-1}\ ere\ i_n & \Longrightarrow & ln\bowtie ess_1 ...\bowtie ess_{n-2}\bowtie ere\bowtie\ [i_1, ..., i_{n-2}, i_{n-1}, i_n] \\ ln\ i_1\ ess_1\ ...\ i_{n-2}\ ess_{n-2}\ i_{n-1}\ sre\ i_n & \Longrightarrow & ln\bowtie ess_1 ...\bowtie ess_{n-2}\bowtie sre\bowtie\ [i_1, ..., i_{n-2}, i_{n-1}, i_n] \end{array}$$

C.4.9.4 `PostfixGenName`

$$\begin{array}{rcl} i\ post & \Longrightarrow & \bowtie post\ [i] \\ i_1 el\ i_2\ ess_2\ ...\ i_{n-1}\ ess_{n-1}\ i_n\ er & \Longrightarrow & \bowtie el\bowtie ess_2 ...\bowtie ess_{n-1}\bowtie er\ [i_1, i_2, ..., i_{n-1}, i_n] \\ i_1 el\ i_2\ ess_2\ ...\ i_{n-1}\ ess_{n-1}\ i_n\ sr & \Longrightarrow & \bowtie el\bowtie ess_2 ...\bowtie ess_{n-1}\bowtie sr\ [i_1, i_2, ..., i_{n-1}, i_n] \end{array}$$

C.4.9.5 `InfixGenName`

$$i_1 in\ i_2 \Longrightarrow \bowtie in\bowtie\ [i_1, i_2]$$

$$i_1 el\ i_2\ ess_2\ ...\ i_{n-2}\ ess_{n-2}\ i_{n-1}\ ere\ i_n \implies \bowtie el \bowtie ess_2 ... \bowtie ess_{n-2} \bowtie ere \bowtie\ [i_1, i_2, ..., i_{n-2}, i_{n-1}, i_n]$$
$$i_1 el\ i_2\ ess_2\ ...\ i_{n-2}\ ess_{n-2}\ i_{n-1}\ sre\ i_n \implies \bowtie el \bowtie ess_2 ... \bowtie ess_{n-2} \bowtie sre \bowtie\ [i_1, i_2, ..., i_{n-2}, i_{n-1}, i_n]$$

C.4.9.6 `NofixGenName`

$$ln\ i_1\ ess_1\ ...\ i_{n-1}\ ess_{n-1}\ i_n\ er \implies ln \bowtie ess_1 ... \bowtie ess_{n-1} \bowtie er\ [i_1, ..., i_{n-1}, i_n]$$
$$ln\ i_1\ ess_1\ ...\ i_{n-1}\ ess_{n-1}\ i_n\ sr \implies ln \bowtie ess_1 ... \bowtie ess_{n-1} \bowtie sr\ [i_1, ..., i_{n-1}, i_n]$$

C.4.10 Free types

C.4.10.1 Syntax

```
Paragraph      = ...
               | Freetype , { & , Freetype } , END
               | ...
               ;

Freetype       = NAME , ::= , Branch , { |-tok , Branch } ;

Branch         = DeclName , [ ⟪ , Expression , ⟫ ] ;
```

C.4.10.2 Transformation

The transformation of free types paragraphs is done in two stages. First, the branches are permuted to bring elements to the front and injections to the rear.

$$...\mid g\langle\!\langle e \rangle\!\rangle \mid h \mid ... \implies ... \mid h \mid g\langle\!\langle e \rangle\!\rangle \mid ...$$

Exhaustive application of this syntactic transformation rule effects a sort.

The second stage requires implicit generic instantiations to have been filled in, which is done during typechecking (section 13.2.3.3). Hence that second stage is delayed until after typechecking, where it appears in the form of a semantic transformation rule (section 14.2.3.1).

C.4.10.3 Type

$$
\frac{\begin{array}{l}
\beta \vdash^{\mathcal{E}} e_{1\,1} \mathrel{⦂} \tau_{1\,1} \quad \ldots \quad \beta \vdash^{\mathcal{E}} e_{1\,n_1} \mathrel{⦂} \tau_{1\,n_1} \\
\vdots \\
\beta \vdash^{\mathcal{E}} e_{r\,1} \mathrel{⦂} \tau_{r\,1} \quad \ldots \quad \beta \vdash^{\mathcal{E}} e_{r\,n_r} \mathrel{⦂} \tau_{r\,n_r}
\end{array}}{\Sigma \vdash^{\mathcal{D}} \begin{array}{l}
f_1 ::= h_{1\,1} \mid \ldots \mid h_{1\,m_1} \mid \\
\quad g_{1\,1} \langle\!\langle e_{1\,1} \mathrel{⦂} \tau_{1\,1} \rangle\!\rangle \mid \\
\quad \vdots \mid \\
\quad g_{1\,n_1} \langle\!\langle e_{1\,n_1} \mathrel{⦂} \tau_{1\,n_1} \rangle\!\rangle \ \& \\
\vdots \ \& \\
f_r ::= h_{r\,1} \mid \ldots \mid h_{r\,m_r} \mid \\
\quad g_{r\,1} \langle\!\langle e_{r\,1} \mathrel{⦂} \tau_{r\,1} \rangle\!\rangle \mid \\
\quad \vdots \mid \\
\quad g_{r\,n_r} \langle\!\langle e_{r\,n_r} \mathrel{⦂} \tau_{r\,n_r} \rangle\!\rangle
\end{array} \quad \mathtt{END} \mathrel{⦂} \sigma}
\left(\begin{array}{l}
\#\{f_1,\ h_{1\,1},\ \ldots,\ h_{1\,m_1},\ g_{1\,1},\ \ldots,\ g_{1\,n_1}, \\
\vdots, \\
\quad f_r,\ h_{r\,1},\ \ldots,\ h_{r\,m_r},\ g_{r\,1},\ \ldots,\ g_{r\,n_r}\} \\
\ = r + m_1 + \ldots + m_r + n_1 + \ldots + n_r \\
\beta = \Sigma \oplus \{f_1 \mapsto \mathbb{P}(\mathtt{GIVEN}\ f_1),\ \ldots,\ f_r \mapsto \mathbb{P}(\mathtt{GIVEN}\ f_r)\} \\
\tau_{1\,1} = \mathbb{P}\,\alpha_{1\,1} \quad \ldots \quad \tau_{1\,n_1} = \mathbb{P}\,\alpha_{1\,n_1} \\
\quad \vdots \\
\tau_{r\,1} = \mathbb{P}\,\alpha_{r\,1} \quad \ldots \quad \tau_{r\,n_r} = \mathbb{P}\,\alpha_{r\,n_r} \\
\sigma \ = \ f_1 : \mathbb{P}(\mathtt{GIVEN}\ f_1); \\
\qquad h_{1\,1} : \mathtt{GIVEN}\ f_1;\ \ldots;\ h_{1\,m_1} : \mathtt{GIVEN}\ f_1; \\
\qquad g_{1\,1} : \mathbb{P}(\tau_{1\,1} \times \mathtt{GIVEN}\ f_1); \\
\qquad \quad \vdots; \\
\qquad g_{1\,n_1} : \mathbb{P}(\tau_{1\,n_1} \times \mathtt{GIVEN}\ f_1); \\
\quad \vdots; \\
\quad f_r : \mathbb{P}(\mathtt{GIVEN}\ f_r); \\
\qquad h_{r\,1} : \mathtt{GIVEN}\ f_r;\ \ldots;\ h_{r\,m_r} : \mathtt{GIVEN}\ f_r; \\
\qquad g_{r\,1} : \mathbb{P}(\tau_{r\,1} \times \mathtt{GIVEN}\ f_r); \\
\qquad \quad \vdots; \\
\qquad g_{r\,n_r} : \mathbb{P}(\tau_{r\,n_r} \times \mathtt{GIVEN}\ f_r)
\end{array}\right)
$$

In a free types paragraph, the names of the free types, elements and injections shall all be different. The expressions representing the domains of the injections are typechecked in an environment overridden by the names of the free types, and shall all be sets. The annotation of the paragraph is the signature whose names are those of all the free types, the elements, and the injections, each associated with the corresponding type.

C.4.10.4 Semantics

A free types paragraph is semantically equivalent to the sequence of given type paragraph and axiomatic definition paragraph defined here.

NOTE 1 This exploits notation that is not present in the annotated syntax for the purpose of abbreviation.

$$
\begin{array}{l}
f_1 ::= h_{1\,1} \mid \ldots \mid h_{1\,m_1} \mid g_{1\,1} \langle\!\langle e_{1\,1} \rangle\!\rangle \mid \ldots \mid g_{1\,n_1} \langle\!\langle e_{1\,n_1} \rangle\!\rangle \\
\& \ \ldots \ \& \\
f_r ::= h_{r\,1} \mid \ldots \mid h_{r\,m_r} \mid g_{r\,1} \langle\!\langle e_{r\,1} \rangle\!\rangle \mid \ldots \mid g_{r\,n_r} \langle\!\langle e_{r\,n_r} \rangle\!\rangle \\
\\
\qquad \Longrightarrow \\
\\
[f_1, \ldots, f_r] \\
\mathtt{END}
\end{array}
$$

$$
\begin{array}{l}
\mathtt{AX} \\
h_{1\,1},\ ...,\ h_{1\,m_1} : f_1 \\
\vdots \\
h_{r\,1},\ ...,\ h_{r\,m_r} : f_r \\
g_{1\,1} : \mathbb{P}(e_{1\,1} \times f_1);\ ...;\ g_{1\,n_1} : \mathbb{P}(e_{1\,n_1} \times f_1) \\
\vdots \\
g_{r\,1} : \mathbb{P}(e_{r\,1} \times f_r);\ ...;\ g_{r\,n_r} : \mathbb{P}(e_{r\,n_r} \times f_r) \\
| \\
(\forall\ u : e_{1\,1} \bullet \exists_1\ x : g_{1\,1} \bullet x\ .\ 1 = u) \land ... \land (\forall\ u : e_{1\,n_1} \bullet \exists_1\ x : g_{1\,n_1} \bullet x\ .\ 1 = u) \\
\vdots \land \\
(\forall\ u : e_{r\,1} \bullet \exists_1\ x : g_{r\,1} \bullet x\ .\ 1 = u) \land ... \land (\forall\ u : e_{r\,n_r} \bullet \exists_1\ x : g_{r\,n_r} \bullet x\ .\ 1 = u) \\
\\
(\forall\ u, v : e_{1\,1} \mid g_{1\,1} u = g_{1\,1} v \bullet u = v) \land ... \land (\forall\ u, v : e_{1\,n_1} \mid g_{1\,n_1} u = g_{1\,n_1} v \bullet u = v) \\
\vdots \land \\
(\forall\ u, v : e_{r\,1} \mid g_{r\,1} u = g_{r\,1} v \bullet u = v) \land ... \land (\forall\ u, v : e_{r\,n_r} \mid g_{r\,n_r} u = g_{r\,n_r} v \bullet u = v) \\
\\
\forall\ b_1, b_2 : \mathbb{N} \bullet \\
\quad (\forall\ w : f_1 \mid \\
\qquad (b_1 = 1 \land w = h_{1\,1} \lor ... \lor b_1 = m_1 \land w = h_{1\,m_1} \lor \\
\qquad\quad b_1 = m_1 + 1 \land w \in \{x : g_{1\,1} \bullet x\ .\ 2\} \lor ... \lor b_1 = m_1 + n_1 \land w \in \{x : g_{1\,n_1} \bullet x\ .\ 2\}) \\
\qquad \land (b_2 = 1 \land w = h_{1\,1} \lor ... \lor b_2 = m_1 \land w = h_{1\,m_1} \lor \\
\qquad\quad b_2 = m_1 + 1 \land w \in \{x : g_{1\,1} \bullet x\ .\ 2\} \lor ... \lor b_2 = m_1 + n_1 \land w \in \{x : g_{1\,n_1} \bullet x\ .\ 2\}) \bullet \\
\qquad\qquad b_1 = b_2) \land \\
\quad \vdots \land \\
\quad (\forall\ w : f_r \mid \\
\qquad (b_1 = 1 \land w = h_{r\,1} \lor ... \lor b_1 = m_r \land w = h_{r\,m_r} \lor \\
\qquad\quad b_1 = m_r + 1 \land w \in \{x : g_{r\,1} \bullet x\ .\ 2\} \lor ... \lor b_1 = m_r + n_r \land w \in \{x : g_{r\,n_r} \bullet x\ .\ 2\}) \\
\qquad \land (b_2 = 1 \land w = h_{r\,1} \lor ... \lor b_2 = m_r \land w = h_{r\,m_r} \lor \\
\qquad\quad b_2 = m_r + 1 \land w \in \{x : g_{r\,1} \bullet x\ .\ 2\} \lor ... \lor b_2 = m_r + n_r \land w \in \{x : g_{r\,n_r} \bullet x\ .\ 2\}) \bullet \\
\qquad\qquad b_1 = b_2) \\
\\
\forall\ w_1 : \mathbb{P}\, f_1;\ ...;\ w_r : \mathbb{P}\, f_r \mid \\
\qquad h_{1\,1} \in w_1 \land ... \land h_{1\,m_1} \in w_1 \land \\
\qquad \vdots \land \\
\qquad h_{r\,1} \in w_r \land ... \land h_{r\,m_r} \in w_r \land \\
\qquad (\forall\ y : (\mu\ f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{1\,1}) \bullet g_{1\,1} y \in w_1) \land \\
\qquad ... \land (\forall\ y : (\mu\ f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{1\,n_1}) \bullet g_{1\,n_1} y \in w_1) \land \\
\qquad \vdots \land \\
\qquad (\forall\ y : (\mu\ f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{r\,1}) \bullet g_{r\,1} y \in w_r) \land \\
\qquad ... \land (\forall\ y : (\mu\ f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{r\,n_r}) \bullet g_{r\,n_r} y \in w_r) \bullet \\
\quad w_1 = f_1 \land ... \land w_r = f_r \\
\mathtt{END}
\end{array}
$$

The type names are introduced by the given types paragraph. The elements are declared as members of their corresponding free types. The injections are declared as functions from values in their domains to their corresponding free type.

The first of the four blank-line separated predicates is the total functionality property. It ensures that for every injection, the injection is functional at every value in its domain.

The second of the four blank-line separated predicates is the injectivity property. It ensures that for every injection, any pair of values in its domain for which the injection returns the same value shall be a pair of equal values (hence the name injection).

The third of the four blank-line separated predicates is the disjointness property. It ensures that for every free type, every pair of values of the free type are equal only if they are the same element or are returned by application of the same injection to equal values.

The fourth of the four blank-line separated predicates is the induction property. It ensures that for every free type, its members are its elements, the values returned by its injections, and nothing else.

The generated μ expressions in the induction property are intended to effect substitutions of all references to the free type names, including any such references within generic instantiation lists in the e expressions.

NOTE 2 That is why this is a semantic transformation not a syntactic one: all implicit generic instantiations shall have been made explicit before it is applied.

NOTE 3 The right-hand side of this transformation could have been expressed using the following notation from the mathematical toolkit, but for the desire to define the core language separately from the mathematical toolkit.

$[f_1, ..., f_r]$
`END`

`AX`
$h_{1\,1},\ ...,\ h_{1\,m_1} : f_1$
$\vdots$
$h_{r\,1},\ ...,\ h_{r\,m_r} : f_r$
$g_{1\,1} : e_{1\,1} \rightarrowtail f_1;\ ...;\ g_{1\,n_1} : e_{1\,n_1} \rightarrowtail f_1$
$\vdots$
$g_{r\,1} : e_{r\,1} \rightarrowtail f_r;\ ...;\ g_{r\,n_r} : e_{r\,n_r} \rightarrowtail f_r$
$|$
$disjoint\langle\{h_{1\,1}\},\ ...,\ \{h_{1\,m_1}\},\ ran\, g_{1\,1},\ ...,\ ran\, g_{1\,n_1}\rangle$
$\vdots$
$disjoint\langle\{h_{r\,1}\},\ ...,\ \{h_{r\,m_r}\},\ ran\, g_{r\,1},\ ...,\ ran\, g_{r\,n_r}\rangle$
$\forall\ w_1 : \mathbb{P}\, f_1;\ ...;\ w_r : \mathbb{P}\, f_r\ |$
$\{h_{1\,1},\ ...,\ h_{1\,m_1}\} \cup g_{1\,1}(\!|\ \mu\, f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{1\,1}\ |\!)$
$\cup ... \cup g_{1\,n_1}(\!|\ \mu\, f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{1\,n_1}\ |\!) \subseteq w_1 \land$
$\vdots \land$
$\{h_{r\,1},\ ...,\ h_{r\,m_r}\} \cup g_{r\,1}(\!|\ \mu\, f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{r\,1}\ |\!)$
$\cup ... \cup g_{r\,n_r}(\!|\ \mu\, f_1 == w_1;\ ...;\ f_r == w_r \bullet e_{r\,n_r}\ |\!) \subseteq w_r \bullet$
$w_1 = f_1 \land ... \land w_r = f_r$
`END`

C.4.11 Conjecture

C.4.11.1 Syntax

```
Paragraph        = ...
                 | ⊢? , Predicate , END
                 | ...
                 ;
```


C.4.11.2 Type

$$\frac{\Sigma \vdash^{\mathcal{P}} p}{\Sigma \vdash^{\mathcal{D}} \vdash ?\ p\ \mathtt{END}\ \mathbin{⨾}\ \sigma}(\sigma = \epsilon)$$

In a conjecture paragraph $\vdash ?\ p$ END, the predicate p shall be well-typed. The annotation of the paragraph is the empty signature.

C.4.11.3 Semantics

The conjecture paragraph $\vdash ?\ p$ END expresses a property that may logically follow from the specification. It may be a starting point for a proof.

$$[\![\vdash ?\ p\ \mathtt{END}]\!]^{\mathcal{D}} = id\ Model$$

It relates a model to itself: the truth of p in a model does not affect the meaning of the specification.

C.4.12 Generic conjecture

C.4.12.1 Syntax

```
Paragraph       = ...
                | [-tok , Formals , ]-tok , ⊢? , Predicate , END
                | ...
                ;
```

C.4.12.2 Type

$$\frac{\Sigma \oplus \{i_1 \mapsto \mathbb{P}(\mathtt{GENTYPE}\ i_1),\ ...,\ i_n \mapsto \mathbb{P}(\mathtt{GENTYPE}\ i_n)\} \vdash^{\mathcal{P}} p}{\Sigma \vdash^{\mathcal{D}} [i_1, ..., i_n] \vdash ?\ p\ \mathtt{END}\ \mathbin{⨾}\ \sigma}\left(\begin{array}{l} \#\{i_1,\ ...,\ i_n\} = n \\ \sigma = \epsilon \end{array}\right)$$

In a generic conjecture paragraph $[i_1, ..., i_n] \vdash ?\ p$ END, there shall be no duplication of names within the generic parameters. The predicate p shall be well-typed in an environment overridden by the generic parameters. The annotation of the paragraph is the empty signature.

C.4.12.3 Semantics

The generic conjecture paragraph $[i_1, ..., i_n] \vdash ?\ p$ END expresses a generic property that may logically follow from the specification. It may be a starting point for a proof.

$$[\![[i_1, ..., i_n] \vdash ?\ p\ \mathtt{END}]\!]^{\mathcal{D}} = id\ Model$$

It relates a model to itself: the truth of p in a model does not affect the meaning of the specification.

C.4.13 Operator template

An operator template has only syntactic significance: it notifies the reader to treat all occurrences in this section of the words in the template, with whatever strokes they are decorated, as particular prefix, infix, postfix or nofix names. The category of the operator—relation, function, or generic—determines how applications of the operator are interpreted— as relational predicates, application expressions, or generic instantiation expressions respectively.

C.4.13.1 Syntax

```
Paragraph        = ...
                 | OperatorTemplate , END
                 ;

OperatorTemplate = relation , Template
                 | function , CategoryTemplate
                 | generic , CategoryTemplate
                 ;

CategoryTemplate = Prec , PrefixTemplate
                 | Prec , PostfixTemplate
                 | Prec , Assoc , InfixTemplate
                 | NofixTemplate
                 ;

Prec             = NUMERAL ;

Assoc            = leftassoc
                 | rightassoc
                 ;

Template         = PrefixTemplate
                 | PostfixTemplate
                 | InfixTemplate
                 | NofixTemplate
                 ;

PrefixTemplate   = (-tok , PrefixName , )-tok ;

PostfixTemplate  = (-tok , PostfixName , )-tok ;

InfixTemplate    = (-tok , InfixName , )-tok ;

NofixTemplate    = (-tok , NofixName , )-tok ;
```

C.5 Predicate

C.5.1 Introduction

A `Predicate` expresses constraints between the values associated with names. A `Predicate` can be any of universal quantification, existential quantification, unique existential quantification, newline conjunction, semicolon conjunction, equivalence, implication, disjunction, conjunction, negation, relation operator application, membership, schema predicate, truth, falsity, or parenthesized predicate.

C.5.2 Universal quantification

C.5.2.1 Syntax

```
Predicate        = ∀ , SchemaText , • , Predicate
                 | ...
                 ;
```

C.5.2.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e \,⨾\, \tau \qquad \Sigma \oplus \beta \vdash^{\mathcal{P}} p}{\Sigma \vdash^{\mathcal{P}} \forall\,(e \,⨾\, \tau) \bullet p}(\tau = \mathbb{P}[\beta])$$

In a universal quantification predicate $\forall\ e \bullet p$, expression e shall be a schema, and predicate p shall be well-typed in the environment overridden by the signature of schema e.

C.5.2.3 Semantics

The universal quantification predicate $\forall\ e \bullet p$ is *true* if and only if predicate p is *true* for all bindings of the schema e.

$$[\![\ \forall\ e \bullet p\ ]\!]^{\mathcal{P}} \quad = \quad \{M : Model \mid \forall\ t : [\![\ e\ ]\!]^{\mathcal{E}} M \bullet M \oplus t \in [\![\ p\ ]\!]^{\mathcal{P}} \bullet M\}$$

In terms of the semantic universe, it is *true* in those models for which p is *true* in that model overridden by all bindings in the semantic value of e, and is *false* otherwise.

C.5.3 Existential quantification

C.5.3.1 Syntax

```
Predicate       =  ...
                |  ∃ , SchemaText , ● , Predicate
                |  ...
                ;
```

C.5.3.2 Transformation

The existential quantification predicate $\exists\ t \bullet p$ is *true* if and only if p is *true* for at least one value of t.

$$\exists\ t \bullet p \quad \Longrightarrow \quad \neg\ \forall\ t \bullet \neg\ p$$

It is semantically equivalent to p being *false* for not all values of t.

C.5.4 Unique existential quantification

C.5.4.1 Syntax

```
Predicate       =  ...
                |  ∃₁ , SchemaText , ● , Predicate
                |  ...
                ;
```

C.5.4.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e \,⨾\, \tau \qquad \Sigma \oplus \beta \vdash^{\mathcal{P}} p}{\Sigma \vdash^{\mathcal{P}} \exists_1\,(e \,⨾\, \tau) \bullet p}(\tau = \mathbb{P}[\beta])$$

In a unique existential quantification predicate $\exists_1\ e \bullet p$, expression e shall be a schema, and predicate p shall be well-typed in the environment overridden by the signature of schema e.

C.5.4.3 Semantics

The unique existential quantification predicate $\exists_1\ e \bullet p$ is *true* if and only if there is exactly one value for e for which p is *true*.

$$\exists_1\ e \bullet p \quad \Longrightarrow \quad \neg\,(\forall\ e \bullet \neg\,(p \land (\forall\,[e \mid p]^{\bowtie} \bullet \theta\, e = \theta\, e^{\ \bowtie})))$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$\exists_1 \; e \bullet p \quad \Longrightarrow \quad \exists \; e \bullet p \wedge (\forall \, [e \mid p]^{\bowtie} \bullet \theta \, e = \theta \, e^{\;\bowtie})$$

It is semantically equivalent to there existing at least one value for e for which p is *true* and all those values for which it is *true* being the same.

C.5.5 Newline conjunction

C.5.5.1 Syntax

```
Predicate        = ...
                 | Predicate , NL , Predicate
                 | ...
                 ;
```

C.5.5.2 Transformation

The newline conjunction predicate p_1 NL p_2 is *true* if and only if both its predicates are *true*.

$$p_1 \; \mathtt{NL} \; p_2 \quad \Longrightarrow \quad p_1 \wedge p_2$$

It is semantically equivalent to the conjunction predicate $p_1 \wedge p_2$.

C.5.6 Semicolon conjunction

C.5.6.1 Syntax

```
Predicate        = ...
                 | Predicate , ;-tok , Predicate
                 | ...
                 ;
```

C.5.6.2 Transformation

The semicolon conjunction predicate $p_1;\; p_2$ is *true* if and only if both its predicates are *true*.

$$p_1; \; p_2 \quad \Longrightarrow \quad p_1 \wedge p_2$$

It is semantically equivalent to the conjunction predicate $p_1 \wedge p_2$.

C.5.7 Equivalence

C.5.7.1 Syntax

```
Predicate        = ...
                 | Predicate , ⇔ , Predicate
                 | ...
                 ;
```


C.5.7.2 Transformation

The equivalence predicate $p_1 \Leftrightarrow p_2$ is *true* if and only if both p_1 and p_2 are *true* or neither is *true*.

$$p_1 \Leftrightarrow p_2 \quad \Longrightarrow \quad (p_1 \Rightarrow p_2) \land (p_2 \Rightarrow p_1)$$

It is semantically equivalent to each of p_1 and p_2 being *true* if the other is *true*.

C.5.8 Implication

C.5.8.1 Syntax

```
Predicate        = ...
                 | Predicate , ⇒ , Predicate
                 | ...
                 ;
```

C.5.8.2 Transformation

The implication predicate $p_1 \Rightarrow p_2$ is *true* if and only if p_2 is *true* if p_1 is *true*.

$$p_1 \Rightarrow p_2 \quad \Longrightarrow \quad \neg\, p_1 \lor p_2$$

It is semantically equivalent to p_1 being *false* disjoined with p_2 being *true*.

C.5.9 Disjunction

C.5.9.1 Syntax

```
Predicate        = ...
                 | Predicate , ∨ , Predicate
                 | ...
                 ;
```

C.5.9.2 Transformation

The disjunction predicate $p_1 \lor p_2$ is *true* if and only if at least one of p_1 and p_2 is *true*.

$$p_1 \lor p_2 \quad \Longrightarrow \quad \neg\,(\neg\, p_1 \land \neg\, p_2)$$

It is semantically equivalent to not both of p_1 and p_2 being *false*.

C.5.10 Conjunction

C.5.10.1 Syntax

```
Predicate        = ...
                 | Predicate , ∧ , Predicate
                 | ...
                 ;
```


C.5.10.2 Type

$$\frac{\Sigma \vdash^{\mathcal{P}} p_1 \qquad \Sigma \vdash^{\mathcal{P}} p_2}{\Sigma \vdash^{\mathcal{P}} p_1 \land p_2}$$

A conjunction predicate $p_1 \land p_2$ is well-typed if and only if predicates p_1 and p_2 are well-typed.

C.5.10.3 Semantics

The conjunction predicate $p_1 \land p_2$ is *true* if and only if p_1 and p_2 are *true*.

$$[\![\, p_1 \land p_2 \,]\!]^{\mathcal{P}} \;=\; [\![\, p_1 \,]\!]^{\mathcal{P}} \cap [\![\, p_2 \,]\!]^{\mathcal{P}}$$

In terms of the semantic universe, it is *true* in those models in which both p_1 and p_2 are *true*, and is *false* otherwise.

C.5.11 Negation

C.5.11.1 Syntax

```
Predicate        = ...
                 | ¬ , Predicate
                 | ...
                 ;
```

C.5.11.2 Type

$$\frac{\Sigma \vdash^{\mathcal{P}} p}{\Sigma \vdash^{\mathcal{P}} \neg\, p}$$

A negation predicate $\neg\, p$ is well-typed if and only if predicate p is well-typed.

C.5.11.3 Semantics

The negation predicate $\neg\, p$ is *true* if and only if p is *false*.

$$[\![\, \neg\, p \,]\!]^{\mathcal{P}} \;=\; \mathit{Model} \setminus [\![\, p \,]\!]^{\mathcal{P}}$$

In terms of the semantic universe, it is *true* in all models except those in which p is *true*.

C.5.12 Relation operator application

C.5.12.1 Syntax

```
Predicate        = ...
                 | Relation
                 | ...
                 ;

Relation         = PrefixRel
                 | PostfixRel
                 | InfixRel
                 | NofixRel
                 ;
```



```
PrefixRel        = PREP , Expression
                 | LP , ExpSep , ( Expression , EREP | ExpressionList , SREP ) , Expression
                 ;

PostfixRel       = Expression , POSTP
                 | Expression , ELP , ExpSep , ( Expression , ERP | ExpressionList , SRP )
                 ;

InfixRel         = Expression , ( ∈ | =-tok | IP ) , Expression
                       { ( ∈ | =-tok | IP ) , Expression }
                 | Expression , ELP , ExpSep ,
                     ( Expression , EREP | ExpressionList , SREP ) , Expression
                 ;

NofixRel         = LP , ExpSep , ( Expression , ERP | ExpressionList , SRP ) ;
```

C.5.12.2 Transformation

All relation operator applications are transformed to annotated membership predicates.

Each relation `NAME` should be one for which there is an operator template paragraph in scope (see 12.2.8).

The left-hand sides of many of these transformation rules involve `ExpSep` phrases: they use *es* metavariables. None of them use *ss* metavariables: in other words, only the `Expression ES` case of `ExpSep` is specified, not the `ExpressionList SS` case. Where the latter case occurs in a specification, the `ExpressionList` shall be transformed by rule 12.2.12 to an expression, and thence a transformation analogous to that specified for the former case can be performed, differing only in that a *ss* appears in the relation name in place of an *es*.

C.5.12.3 `PrefixRel`

$$\begin{array}{rcl} prep\ e & \Longrightarrow & e \in prep\bowtie \\ lp\ e_1\ es_1\ ...\ e_{n-2}\ es_{n-2}\ e_{n-1}\ erep\ e_n & \Longrightarrow & (e_1, ..., e_{n-2}, e_{n-1}, e_n) \in lp\bowtie es_1 ... \bowtie es_{n-2} \bowtie erep\bowtie \\ lp\ e_1\ es_1\ ...\ e_{n-2}\ es_{n-2}\ al_{n-1}\ srep\ e_n & \Longrightarrow & (e_1, ..., e_{n-2}, al_{n-1}, e_n) \in lp\bowtie es_1 ... \bowtie es_{n-2} \bowtie srep\bowtie \end{array}$$

C.5.12.4 `PostfixRel`

$$\begin{array}{rcl} e\ postp & \Longrightarrow & e \in \bowtie postp \\ e_1\ elp\ e_2\ es_2\ ...\ e_{n-1}\ es_{n-1}\ e_n\ erp & \Longrightarrow & (e_1, e_2, ..., e_{n-1}, e_n) \in \bowtie elp\bowtie es_2 ... \bowtie es_{n-1} \bowtie erp \\ e_1\ elp\ e_2\ es_2\ ...\ e_{n-1}\ es_{n-1}\ al_n\ srp & \Longrightarrow & (e_1, e_2, ..., e_{n-1}, al_n) \in \bowtie elp\bowtie es_2 ... \bowtie es_{n-1} \bowtie srp \end{array}$$

C.5.12.5 `InfixRel`

$$e_1\ ip_1\ e_2\ ip_2\ e_3\ ... \quad \Longrightarrow \quad e_1\ ip_1\ e_2 \mathbin{⦂} \tau_1 \wedge e_2 \mathbin{⦂} \tau_1\ ip_2\ e_3 \mathbin{⦂} \tau_2\ ...$$

The chained relation $e_1\ ip_1\ e_2\ ip_2\ e_3\ ...$ is semantically equivalent to a conjunction of relational predicates, with the constraint that duplicated expressions be of the same type.

$$\begin{array}{rcl} e_1 = e_2 & \Longrightarrow & e_1 \in \{e_2\} \\ e_1\ ip\ e_2 & \Longrightarrow & (e_1, e_2) \in \bowtie ip\bowtie \end{array}$$

ip in the above transformation is excluded from being $\in$ or $=$, whereas $ip_1, ip_2, ...$ can be $\in$ or $=$.

$$
\begin{array}{rcl}
e_1\ elp\ e_2\ es_2\ ...\ e_{n-2}\ es_{n-2}\ e_{n-1}\ erep\ e_n & \implies & (e_1, e_2, ..., e_{n-2}, e_{n-1}, e_n) \in \bowtie elp \bowtie es_2 ... \bowtie es_{n-2} \bowtie erep \bowtie \\
e_1\ elp\ e_2\ es_2\ ...\ e_{n-2}\ es_{n-2}\ al_{n-1}\ srep\ e_n & \implies & (e_1, e_2, ..., e_{n-2}, al_{n-1}, e_n) \in \bowtie elp \bowtie es_2 ... \bowtie es_{n-2} \bowtie srep \bowtie
\end{array}
$$

C.5.12.6 `NofixRel`

$$
\begin{array}{rcl}
lp\ e_1\ es_1\ ...\ e_{n-1}\ es_{n-1}\ e_n\ erp & \implies & (e_1, ..., e_{n-1}, e_n) \in lp \bowtie es_1 ... \bowtie es_{n-1} \bowtie erp \\
lp\ e_1\ es_1\ ...\ e_{n-1}\ es_{n-1}\ al_n\ srp & \implies & (e_1, ..., e_{n-1}, al_n) \in lp \bowtie es_1 ... \bowtie es_{n-1} \bowtie srp
\end{array}
$$

C.5.12.7 Type

$$
\frac{\Sigma \vdash^{\mathcal{E}} e_1 \mathbin{⦂} \tau_1 \qquad \Sigma \vdash^{\mathcal{E}} e_2 \mathbin{⦂} \tau_2}{\Sigma \vdash^{\mathcal{P}} (e_1 \mathbin{⦂} \tau_1) \in (e_2 \mathbin{⦂} \tau_2)} \left(\tau_2 = \mathbb{P}\, \tau_1 \right)
$$

In a membership predicate $e_1 \in e_2$, expression e_2 shall be a set, and expression e_1 shall be of the same type as the members of set e_2.

C.5.12.8 Semantics

The membership predicate $e_1 \in e_2$ is *true* if and only if the value of e_1 is in the set that is the value of e_2.

$$
[\![\ e_1 \in e_2\ ]\!]^{\mathcal{P}} \ = \ \{M : Model \mid [\![\ e_1\ ]\!]^{\mathcal{E}} M \in [\![\ e_2\ ]\!]^{\mathcal{E}} M \bullet M\}
$$

In terms of the semantic universe, it is *true* in those models in which the semantic value of e_1 is in the semantic value of e_2, and is *false* otherwise.

C.5.13 Schema

C.5.13.1 Syntax

```
Predicate        = ...
                 | Expression
                 | ...
                 ;
```

C.5.13.2 Transformation

The schema predicate e is *true* if and only if the binding of the names in the signature of schema e satisfies the constraints of that schema.

$$
e \implies \theta\, e \in e
$$

It is semantically equivalent to the binding constructed by $\theta\, e$ being a member of the set denoted by schema e.

C.5.14 Truth

C.5.14.1 Syntax

```
Predicate        =  ...
                 |  true
                 |  ...
                 ;
```

C.5.14.2 Type

$$\frac{}{\Sigma \vdash^{\mathcal{P}} \text{true}}$$

A truth predicate is always well-typed.

C.5.14.3 Semantics

A truth predicate is always *true*.

$$[\![\text{ true }]\!]^{\mathcal{P}} = \mathit{Model}$$

In terms of the semantic universe, it is *true* in all models.

C.5.15 Falsity

C.5.15.1 Syntax

```
Predicate        =  ...
                 |  false
                 |  ...
                 ;
```

C.5.15.2 Transformation

The falsity predicate false is never *true*.

$$\text{false} \implies \neg \text{ true}$$

It is semantically equivalent to the negation of true.

C.5.16 Parenthesized predicate

C.5.16.1 Syntax

```
Predicate        =  ...
                 |  (-tok , Predicate , )-tok
                 ;
```

C.5.16.2 Transformation

The parenthesized predicate (p) is *true* if and only if p is *true*.

$$(p) \implies p$$

It is semantically equivalent to p.

C.6 Expression

C.6.1 Introduction

An `Expression` denotes a value in terms of the names with which values are associated by a model. An `Expression` can be any of schema universal quantification, schema existential quantification, schema unique existential quantification, function construction, definite description, substitution expression, schema equivalence, schema implication, schema disjunction, schema conjunction, schema negation, conditional, schema composition, schema piping, schema hiding, schema projection, schema precondition, Cartesian product, powerset, function and generic operator application, application, schema decoration, schema renaming, binding selection, tuple selection, binding construction, reference, generic instantiation, number literal, set extension, set comprehension, characteristic set comprehension, schema construction, binding extension, tuple extension, characteristic definite description, or parenthesized expression.

C.6.2 Schema universal quantification

C.6.2.1 Syntax

```
Expression        = ∀ , SchemaText , • , Expression
                  | ...
                  ;
```

C.6.2.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \mathbin{⨾} \tau_1 \qquad \Sigma \oplus \beta_1 \vdash^{\mathcal{E}} e_2 \mathbin{⨾} \tau_2}{\Sigma \vdash^{\mathcal{E}} \forall (e_1 \mathbin{⨾} \tau_1) \bullet (e_2 \mathbin{⨾} \tau_2) \mathbin{⨾} \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta_1] \\ \tau_2 = \mathbb{P}[\beta_2] \\ \beta_1 \approx \beta_2 \\ \tau_3 = \mathbb{P}[dom\ \beta_1 \lhd \beta_2] \end{array} \right)$$

In a schema universal quantification expression $\forall\ e_1 \bullet e_2$, expression e_1 shall be a schema, and expression e_2, in an environment overridden by the signature of schema e_1, shall also be a schema, and the signatures of these two schemas shall be compatible. The type of the whole expression is that of a schema whose signature is computed by subtracting from the signature of e_2 those pairs whose names are in the signature of e_1.

C.6.2.3 Semantics

The value of the schema universal quantification expression $\forall\ e_1 \bullet e_2$ is the set of bindings of schema e_2 restricted to exclude names that are in the signature of e_1, for all bindings of the schema e_1.

$$[\![\ \forall\ e_1 \bullet e_2 \mathbin{⨾} \mathbb{P}\,\tau\ ]\!]^{\mathcal{E}} \ = \ \lambda\ M : Model \bullet \{t_2 : [\![\ \tau\ ]\!]^{\mathcal{T}} M \mid \forall\ t_1 : [\![\ e_1\ ]\!]^{\mathcal{E}} M \bullet t_1 \cup t_2 \in [\![\ e_2\ ]\!]^{\mathcal{E}}\ (M \oplus t_1) \bullet t_2\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of the bindings (sets of pairs) in the semantic values of the carrier set of the type of the entire schema universal quantification expression in M, for which the union of the bindings (sets of pairs) in e_1 and in the whole expression is in the set that is the semantic value of e_2 in the model M overridden with the binding in e_1.

C.6.3 Schema existential quantification

C.6.3.1 Syntax

```
Expression        = ...
                  | ∃ , SchemaText , • , Expression
                  | ...
                  ;
```

C.6.3.2 Transformation

The value of the schema existential quantification expression $\exists\ t \bullet e$ is the set of bindings of schema e restricted to exclude names that are in the signature of t, for at least one binding of the schema t.

$$\exists\ t \bullet e \quad \Longrightarrow \quad \neg\ \forall\ t \bullet \neg\ e$$

It is semantically equivalent to the result of applying de Morgan's law.

C.6.4 Schema unique existential quantification

C.6.4.1 Syntax

```
Expression        = ...
                  | ∃₁ , SchemaText , • , Expression
                  | ...
                  ;
```

C.6.4.2 Type

$$\frac{\Sigma\ \vdash^{\mathcal{E}}\ e_1\ ⨾\ \tau_1 \qquad \Sigma \oplus \beta_1\ \vdash^{\mathcal{E}}\ e_2\ ⨾\ \tau_2}{\Sigma\ \vdash^{\mathcal{E}}\ \exists_1\ (e_1\ ⨾\ \tau_1) \bullet (e_2\ ⨾\ \tau_2)\ ⨾\ \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta_1] \\ \tau_2 = \mathbb{P}[\beta_2] \\ \beta_1 \approx \beta_2 \\ \tau_3 = \mathbb{P}[dom\ \beta_1 \lhd \beta_2] \end{array}\right)$$

In a schema unique existential quantification expression $\exists_1\ e_1 \bullet e_2$, expression e_1 shall be a schema, and expression e_2, in an environment overridden by the signature of schema e_1, shall also be a schema, and the signatures of these two schemas shall be compatible. The type of the whole expression is that of a schema whose signature is computed by subtracting from the signature of e_2 those pairs whose names are in the signature of e_1.

C.6.4.3 Semantics

The value of the schema unique existential quantification expression $\exists_1\ e_1 \bullet e_2$ is the set of bindings of schema e_2 restricted to exclude names that are in the signature of e_1, for at least one binding of the schema e_1.

$$\exists_1\ e_1 \bullet e_2 \quad \Longrightarrow \quad \neg\ (\forall\ e_1 \bullet \neg\ (e_2 \land (\forall\ [e_1 \mid e_2]^{\bowtie} \bullet \theta\ e_1 = \theta\ e_1{}^{\bowtie})))$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$\exists_1\ e_1 \bullet e_2 \quad \Longrightarrow \quad \exists\ e_1 \bullet e_2 \land (\forall\ [e_1 \mid e_2]^{\bowtie} \bullet \theta\ e_1 = \theta\ e_1{}^{\bowtie})$$

It is semantically equivalent to a schema existential quantification expression, analogous to the unique existential quantification predicate transformation.

C.6.5 Function construction

C.6.5.1 Syntax

```
Expression        = ...
                  | λ , SchemaText , • , Expression
                  | ...
                  ;
```


C.6.5.2 Transformation

The value of the function construction expression $\lambda\ t \bullet e$ is the function associating values of the characteristic tuple of t with corresponding values of e.

$$\lambda\ t \bullet e \quad \Longrightarrow \quad \{t \bullet (\mathit{chartuple}\ t, e)\}$$

It is semantically equivalent to the set of pairs representation of that function.

C.6.6 Definite description

C.6.6.1 Syntax

```
Expression        = ...
                  | μ , SchemaText , • , Expression
                  | ...
                  ;
```

C.6.6.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \mathbin{⨾} \tau_1 \qquad \Sigma \oplus \beta \vdash^{\mathcal{E}} e_2 \mathbin{⨾} \tau_2}{\Sigma \vdash^{\mathcal{E}} \mu\,(e_1 \mathbin{⨾} \tau_1) \bullet (e_2 \mathbin{⨾} \tau_2) \mathbin{⨾} \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \tau_3 = \tau_2 \end{array}\right)$$

In a definite description expression $\mu\ e_1 \bullet e_2$, expression e_1 shall be a schema. The type of the whole expression is the type of expression e_2, as determined in an environment overridden by the signature of schema e_1.

C.6.6.3 Semantics

The value of the definite description expression $\mu\ e_1 \bullet e_2$ is the sole value of e_2 that arises whichever binding is chosen from the set that is the value of schema e_1.

$$\begin{array}{l} \{M : \mathit{Model};\ t_1 : \mathbb{W} \\ \quad \mid t_1 \in [\![\ e_1\ ]\!]^{\mathcal{E}} M \\ \quad \wedge (\forall\ t_3 : [\![\ e_1\ ]\!]^{\mathcal{E}} M \bullet [\![\ e_2\ ]\!]^{\mathcal{E}}\ (M \oplus t_3) = [\![\ e_2\ ]\!]^{\mathcal{E}}\ (M \oplus t_1)) \\ \bullet\ M \mapsto [\![\ e_2\ ]\!]^{\mathcal{E}}\ (M \oplus t_1)\} \qquad \subseteq \quad [\![\ \mu\ e_1 \bullet e_2\ ]\!]^{\mathcal{E}} \end{array}$$

In terms of the semantic universe, its semantic value, given a model M in which the value of e_2 in that model overridden by a binding of the schema e_1 is the same regardless of which binding is chosen, is that value of e_2. In other models, it has a semantic value, but this loose definition of the semantics does not say what it is.

C.6.7 Substitution expression

C.6.7.1 Syntax

```
Expression        = ...
                  | let , DeclName , == , Expression
                        { ;-tok , DeclName , == , Expression }
                            , • , Expression
                  | ...
                  ;
```

C.6.7.2 Transformation

The value of the substitution expression let $i_1 == e_1;\ ...;\ i_n == e_n \bullet e$ is the value of e when all of its references to the names have been substituted by the values of the corresponding expressions.

$$\mathrm{let}\, i_1 == e_1;\ ...;\ i_n == e_n \bullet e \quad \Longrightarrow \quad \mu\ i_1 == e_1;\ ...;\ i_n == i_n \bullet e$$

It is semantically equivalent to the similar definite description expression.

C.6.8 Schema equivalence

C.6.8.1 Syntax

```
Expression        = ...
                  | Expression , ⇔ , Expression
                  | ...
                  ;
```

C.6.8.2 Transformation

The value of the schema equivalence expression $e_1 \Leftrightarrow e_2$ is that schema whose signature is the union of those of schemas e_1 and e_2, and whose bindings are those whose relevant restrictions are either both or neither in e_1 and e_2.

$$e_1 \Leftrightarrow e_2 \quad \Longrightarrow \quad (e_1 \Rightarrow e_2) \land (e_2 \Rightarrow e_1)$$

It is semantically equivalent to the schema conjunction shown above.

C.6.9 Schema implication

C.6.9.1 Syntax

```
Expression        = ...
                  | Expression , ⇒ , Expression
                  | ...
                  ;
```

C.6.9.2 Transformation

The value of the schema implication expression $e_1 \Rightarrow e_2$ is that schema whose signature is the union of those of schemas e_1 and e_2, and whose bindings are those whose restriction to the signature of e_2 is in the value of e_2 if its restriction to the signature of e_1 is in the value of e_1.

$$e_1 \Rightarrow e_2 \quad \Longrightarrow \quad \neg\, e_1 \lor e_2$$

It is semantically equivalent to the schema disjunction shown above.

C.6.10 Schema disjunction

C.6.10.1 Syntax

```
Expression      = ...
                | Expression , ∨ , Expression
                | ...
                ;
```

C.6.10.2 Transformation

The value of the schema disjunction expression $e_1 \lor e_2$ is that schema whose signature is the union of those of schemas e_1 and e_2, and whose bindings are those whose restriction to the signature of e_1 is in the value of e_1 or its restriction to the signature of e_2 is in the value of e_2.

$$e_1 \lor e_2 \quad \Longrightarrow \quad \neg\,(\neg\, e_1 \land \neg\, e_2)$$

It is semantically equivalent to the schema negation shown above.

C.6.11 Schema conjunction

C.6.11.1 Syntax

```
Expression      = ...
                | Expression , ∧ , Expression
                | ...
                ;
```

C.6.11.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \mathbin{⨾} \tau_1 \qquad \Sigma \vdash^{\mathcal{E}} e_2 \mathbin{⨾} \tau_2}{\Sigma \vdash^{\mathcal{E}} (e_1 \mathbin{⨾} \tau_1) \land (e_2 \mathbin{⨾} \tau_2) \mathbin{⨾} \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta_1] \\ \tau_2 = \mathbb{P}[\beta_2] \\ \beta_1 \approx \beta_2 \\ \tau_3 = \mathbb{P}[\beta_1 \cup \beta_2] \end{array}\right)$$

In a schema conjunction expression $e_1 \land e_2$, expressions e_1 and e_2 shall be schemas, and their signatures shall be compatible. The type of the whole expression is that of the schema whose signature is the union of those of expressions e_1 and e_2.

C.6.11.3 Semantics

The value of the schema conjunction expression $e_1 \land e_2$ is the schema resulting from merging the signatures of schemas e_1 and e_2 and conjoining their constraints.

$$[\![\, e_1 \land e_2 \mathbin{⨾} \mathbb{P}\,\tau \,]\!]^{\mathcal{E}} \;=\; \lambda\, M : \mathit{Model} \bullet \{t : [\![\, \tau \,]\!]^{\mathcal{T}} M;\; t_1 : [\![\, e_1 \,]\!]^{\mathcal{E}} M;\; t_2 : [\![\, e_2 \,]\!]^{\mathcal{E}} M \mid t_1 \cup t_2 = t \bullet t\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of the unions of the bindings (sets of pairs) in the semantic values of e_1 and e_2 in M.

C.6.12 Schema negation

C.6.12.1 Syntax

```
Expression      = ...
                | ¬ , Expression
                | ...
                ;
```

C.6.12.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathbin{⨾} \tau_1}{\Sigma \vdash^{\mathcal{E}} \neg\,(e \mathbin{⨾} \tau_1) \mathbin{⨾} \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \tau_2 = \tau_1 \end{array}\right)$$

In a schema negation expression $\neg\, e$, expression e shall be a schema. The type of the whole expression is the same as the type of expression e.

C.6.12.3 Semantics

The value of the schema negation expression $\neg\, e$ is that set of bindings that are of the same type as those in schema e but that are not in schema e.

$$[\![\, \neg\, e \mathbin{⨾} \mathbb{P}\,\tau \,]\!]^{\mathcal{E}} \;=\; \lambda\, M : \mathit{Model} \bullet \{t : [\![\, \tau \,]\!]^{\mathcal{T}} M \mid \neg\, t \in [\![\, e \,]\!]^{\mathcal{E}} M \bullet t\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of the bindings (sets of pairs) that are members of the semantic value of the carrier set of schema e in M such that those bindings are not members of the semantic value of schema e in M.

C.6.13 Conditional

C.6.13.1 Syntax

```
Expression        = ...
                  | if , Predicate , then , Expression , else , Expression
                  | ...
                  ;
```

C.6.13.2 Transformation

The value of the conditional expression $\mathrm{if}\ p\ \mathrm{then}\ e_1\ \mathrm{else}\ e_2$ is the value of e_1 if p is *true*, and is the value of e_2 if p is *false*.

$$\mathrm{if}\ p\ \mathrm{then}\ e_1\ \mathrm{else}\ e_2 \quad\Longrightarrow\quad \mu\ i : \{e_1, e_2\} \mid p \wedge i = e_1 \vee \neg\, p \wedge i = e_2 \bullet i$$

It is semantically equivalent to the definite description expression whose value is either that of e_1 or that of e_2 such that if p is *true* then it is e_1 or if p is *false* then it is e_2.

C.6.14 Schema composition

C.6.14.1 Syntax

```
Expression        = ...
                  | Expression , ⨾ , Expression
                  | ...
                  ;
```


C.6.14.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \mathbin{⦂} \tau_1 \qquad \Sigma \vdash^{\mathcal{E}} e_2 \mathbin{⦂} \tau_2}{\Sigma \vdash^{\mathcal{E}} (e_1 \mathbin{⦂} \tau_1) \mathbin{⨾} (e_2 \mathbin{⦂} \tau_2) \mathbin{⦂} \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta_1] \\ \tau_2 = \mathbb{P}[\beta_2] \\ match = \{i : dom\ \beta_2 \mid i\ decor\ {}' \in dom\ \beta_1 \bullet i\} \\ \beta_3 = \{i : match \bullet i\ decor\ {}'\} \lhd \beta_1 \\ \beta_4 = match \lhd \beta_2 \\ \beta_3 \approx \beta_4 \\ \{i : match \bullet i \mapsto \beta_1(i\ decor\ {}')\} \approx \{i : match \bullet i \mapsto \beta_2\ i\} \\ \tau_3 = \mathbb{P}[\beta_3 \cup \beta_4] \end{array} \right)$$

In a schema composition expression $e_1 \mathbin{⨾} e_2$, expressions e_1 and e_2 shall be schemas. Let *match* be the set of names in schema e_2 for which there are matching primed names in schema e_1. Let β_3 be the signature formed from the components of e_1 excluding the matched primed components. Let β_4 be the signature formed from the components of e_2 excluding the matched unprimed components. Signatures β_3 and β_4 shall be compatible. The types of the excluded matched pairs of components shall be the same. The type of the whole expression is that of a schema whose signature is the union of β_3 and β_4.

C.6.14.3 Semantics

The value of the schema composition expression $e_1 \mathbin{⨾} e_2$ is that schema representing the operation of doing the operations represented by schemas e_1 and e_2 in sequence.

$$\begin{array}{l} \qquad\qquad (e_1 \mathbin{⦂} \mathbb{P}[\sigma_1]) \mathbin{⨾} (e_2 \mathbin{⦂} \mathbb{P}[\sigma_2]) \mathbin{⦂} \mathbb{P}[\sigma] \\ \qquad\qquad\qquad\qquad \Longrightarrow \\ \neg\,(\forall\ e^{\bowtie} \bullet \neg\,(\neg\,(\forall\ e_3 \bullet \neg\,[e_1;\ e^{\bowtie} \mid \theta\, e_3 = \theta\, e^{\bowtie}]) \\ \qquad\qquad \wedge \neg\,(\forall\ e_4 \bullet \neg\,[e_2;\ e^{\bowtie} \mid \theta\, e_4 = \theta\, e^{\bowtie}]))) \\ where\ e_3 == carrier\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i\ decor\ {}' \mapsto \tau \in \sigma_1 \bullet i\ decor\ {}' \mapsto \tau\}] \\ \quad and\ e_4 == carrier\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i \mapsto \tau \in \sigma_2 \bullet i \mapsto \tau\}] \\ \quad and\ e^{\bowtie} == (e_4)^{\bowtie} \end{array}$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$\begin{array}{l} \qquad\qquad (e_1 \mathbin{⦂} \mathbb{P}[\sigma_1]) \mathbin{⨾} (e_2 \mathbin{⦂} \mathbb{P}[\sigma_2]) \mathbin{⦂} \mathbb{P}[\sigma] \\ \qquad\qquad\qquad\qquad \Longrightarrow \\ \exists\ e^{\bowtie} \bullet (\exists\ e_3 \bullet [e_1;\ e^{\bowtie} \mid \theta\, e_3 = \theta\, e^{\bowtie}]) \\ \qquad \wedge (\exists\ e_4 \bullet [e_2;\ e^{\bowtie} \mid \theta\, e_4 = \theta\, e^{\bowtie}]) \\ where\ e_3 == carrier\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i\ decor\ {}' \mapsto \tau \in \sigma_1 \bullet i\ decor\ {}' \mapsto \tau\}] \\ \quad and\ e_4 == carrier\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i \mapsto \tau \in \sigma_2 \bullet i \mapsto \tau\}] \\ \quad and\ e^{\bowtie} == (e_4)^{\bowtie} \end{array}$$

It is semantically equivalent to the existential quantification of the matched pairs of primed components of e_1 and unprimed components of e_2, with those matched pairs being equated.

C.6.15 Schema piping

C.6.15.1 Syntax

```
Expression      = ...
                | Expression , ≫ , Expression
                | ...
                ;
```

C.6.15.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \,⦂\, \tau_1 \qquad \Sigma \vdash^{\mathcal{E}} e_2 \,⦂\, \tau_2}{\Sigma \vdash^{\mathcal{E}} (e_1 \,⦂\, \tau_1) \gg (e_2 \,⦂\, \tau_2) \,⦂\, \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta_1] \\ \tau_2 = \mathbb{P}[\beta_2] \\ match = \{i : \mathtt{NAME} \mid i\ decor\ ! \in dom\ \beta_1 \land i\ decor\ ? \in dom\ \beta_2 \bullet i\} \\ \beta_3 = \{i : match \bullet i\ decor\ !\} \mathbin{⩤} \beta_1 \\ \beta_4 = \{i : match \bullet i\ decor\ ?\} \mathbin{⩤} \beta_2 \\ \beta_3 \approx \beta_4 \\ \{i : match \bullet i \mapsto \beta_1(i\ decor\ !)\} \approx \{i : match \bullet i \mapsto \beta_2\ (i\ decor\ ?)\} \\ \tau_3 = \mathbb{P}[\beta_3 \cup \beta_4] \end{array}\right)$$

In a schema piping expression $e_1 \gg e_2$, expressions e_1 and e_2 shall be schemas. Let *match* be the set of names for which there are matching shrieked names in schema e_1 and queried names in schema e_2. Let β_3 be the signature formed from the components of e_1 excluding the matched shrieked components. Let β_4 be the signature formed from the components of e_2 excluding the matched queried components. Signatures β_3 and β_4 shall be compatible. The types of the excluded matched pairs of components shall be the same. The type of the whole expression is that of a schema whose signature is the union of β_3 and β_4.

C.6.15.3 Semantics

The value of the schema piping expression $e_1 \gg e_2$ is that schema representing the operation formed from the two operations represented by schemas e_1 and e_2 with the outputs of e_1 identified with the inputs of e_2.

$$\begin{array}{l} \qquad\qquad (e_1 \,⦂\, \mathbb{P}[\sigma_1]) \gg (e_2 \,⦂\, \mathbb{P}[\sigma_2]) \,⦂\, \mathbb{P}[\sigma] \\ \qquad\qquad\qquad\qquad \Longrightarrow \\ \neg\ (\forall\ e^{\bowtie} \bullet \neg\ (\neg\ (\forall\ e_3 \bullet \neg\ [e_1;\ e^{\bowtie} \mid \theta\ e_3 = \theta\ e^{\bowtie}]) \\ \qquad \land \neg\ (\forall\ e_4 \bullet \neg\ [e_2;\ e^{\bowtie} \mid \theta\ e_4 = \theta\ e^{\bowtie}]))) \\ where\ e_3 == carrier\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i\ decor\ ! \mapsto \tau \in \sigma_1 \bullet i\ decor\ ! \mapsto \tau\}] \\ \quad and\ e_4 == carrier\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i\ decor\ ? \mapsto \tau \in \sigma_2 \bullet i\ decor\ ? \mapsto \tau\}] \\ \quad and\ e^{\bowtie} == (e_4)^{\bowtie} \end{array}$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$\begin{array}{l} \qquad\qquad (e_1 \,⦂\, \mathbb{P}[\sigma_1]) \gg (e_2 \,⦂\, \mathbb{P}[\sigma_2]) \,⦂\, \mathbb{P}[\sigma] \\ \qquad\qquad\qquad\qquad \Longrightarrow \\ \exists\ e^{\bowtie} \bullet (\exists\ e_3 \bullet [e_1;\ e^{\bowtie} \mid \theta\ e_3 = \theta\ e^{\bowtie}]) \\ \qquad \land (\exists\ e_4 \bullet [e_2;\ e^{\bowtie} \mid \theta\ e_4 = \theta\ e^{\bowtie}]) \\ where\ e_3 == carrier\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i\ decor\ ! \mapsto \tau \in \sigma_1 \bullet i\ decor\ ! \mapsto \tau\}] \\ \quad and\ e_4 == carrier\ \ [\{i : \mathtt{NAME};\ \tau : \mathtt{Type} \mid i\ decor\ ? \mapsto \tau \in \sigma_2 \bullet i\ decor\ ? \mapsto \tau\}] \\ \quad and\ e^{\bowtie} == (e_4)^{\bowtie} \end{array}$$

It is semantically equivalent to the existential quantification of the matched pairs of shrieked components of e_1 and queried components of e_2, with those matched pairs being equated.

C.6.16 Schema hiding

C.6.16.1 Syntax

```
Expression      = ...
                | Expression , \ , (-tok , DeclName , { ,-tok , DeclName } , )-tok
                | ...
                ;
```


C.6.16.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e \,⦂\, \tau_1}{\Sigma \vdash^{\mathcal{E}} (e \,⦂\, \tau_1) \setminus (i_1, ..., i_n) \,⦂\, \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \{i_1,\ ...,\ i_n\} \subseteq dom\ \beta \\ \tau_2 = \mathbb{P}[\{i_1,\ ...,\ i_n\} \lhd \beta] \end{array} \right)$$

In a schema hiding expression $e \setminus (i_1, ..., i_n)$, expression e shall be a schema, and the names to be hidden shall all be in the signature of that schema. The type of the whole expression is that of a schema whose signature is computed by subtracting from the signature of expression e those pairs whose names are hidden.

C.6.16.3 Semantics

The value of the schema hiding expression $e \setminus (i_1, ..., i_n)$ is that schema whose signature is that of schema e minus the hidden names, and whose bindings have the same values as those in schema e.

$$(e \,⦂\, \mathbb{P}[\sigma]) \setminus (i_1, ..., i_n) \quad \Longrightarrow \quad \neg\, (\forall\ i_1 : carrier\ (\sigma\ i_1);\ ...;\ i_n : carrier\ (\sigma\ i_n) \bullet \neg\, e)$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$(e \,⦂\, \mathbb{P}[\sigma]) \setminus (i_1, ..., i_n) \quad \Longrightarrow \quad \exists\ i_1 : carrier\ (\sigma\ i_1);\ ...;\ i_n : carrier\ (\sigma\ i_n) \bullet e$$

It is semantically equivalent to the schema existential quantification of the hidden names $i_1, ..., i_n$ from the schema e.

C.6.17 Schema projection

C.6.17.1 Syntax

```
Expression      = ...
                | Expression , ↾ , Expression
                | ...
                ;
```

C.6.17.2 Transformation

The value of the schema projection expression $e_1 \upharpoonright e_2$ is the schema that is like the conjunction $e_1 \land e_2$ but whose signature is restricted to just that of schema e_2.

$$e_1 \upharpoonright e_2 \quad \Longrightarrow \quad \{e_1;\ e_2 \bullet \theta\, e_2\}$$

It is semantically equivalent to that set of bindings of names in the signature of e_2 to values that satisfy the constraints of both e_1 and e_2.

C.6.18 Schema precondition

C.6.18.1 Syntax

```
Expression      = ...
                | pre , Expression
                | ...
                ;
```


C.6.18.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathbin{⨾} \tau_1}{\Sigma \vdash^{\mathcal{E}} \text{pre}\ (e \mathbin{⨾} \tau_1) \mathbin{⨾} \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \tau_2 = \mathbb{P}[\{i, j : \mathtt{NAME} \mid j \in dom\ \beta \land (j = i\ decor\ ' \lor j = i\ decor\ !) \bullet j\} \mathbin{⩤} \beta] \end{array} \right)$$

In a schema precondition expression pre e, expression e shall be a schema. The type of the whole expression is that of a schema whose signature is computed by subtracting from the signature of e those pairs whose names have primed or shrieked decorations.

C.6.18.3 Semantics

The value of the schema precondition expression pre e is that schema which is like schema e but without its primed and shrieked components.

$$\text{pre}(e \mathbin{⨾} \mathbb{P}[\sigma_1]) \mathbin{⨾} \mathbb{P}[\sigma_2] \quad \Longrightarrow \quad \neg\, (\forall\ carrier\ [\sigma_1 \setminus \sigma_2] \bullet \neg\, e)$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$\text{pre}(e \mathbin{⨾} \mathbb{P}[\sigma_1]) \mathbin{⨾} \mathbb{P}[\sigma_2] \quad \Longrightarrow \quad \exists\ carrier\ [\sigma_1 \setminus \sigma_2] \bullet e$$

It is semantically equivalent to the existential quantification of the primed and shrieked components from the schema e.

C.6.19 Cartesian product

C.6.19.1 Syntax

```
Expression        = ...
                  | Expression , × , Expression , { × , Expression }
                  | ...
                  ;
```

C.6.19.2 Transformation

The value of the Cartesian product expression $e_1 \times ... \times e_n$ is the set of all tuples whose components are members of the corresponding sets that are the values of its expressions.

$$e_1 \times ... \times e_n \quad \Longrightarrow \quad \{i_1 : e_1;\ ...;\ i_n : e_n \bullet (i_1, ..., i_n)\}$$

It is semantically equivalent to the set comprehension expression that declares members of the sets and assembles those members into tuples.

C.6.20 Powerset

C.6.20.1 Syntax

```
Expression        = ...
                  | ℙ , Expression
                  | ...
                  ;
```


C.6.20.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e \,⦂\, \tau_1}{\Sigma \vdash^{\mathcal{E}} \mathbb{P}(e \,⦂\, \tau_1) \,⦂\, \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}\,\alpha \\ \tau_2 = \mathbb{P}\,\tau_1 \end{array} \right)$$

In a powerset expression $\mathbb{P}\,e$, expression e shall be a set. The type of the whole expression is then a powerset of the type of expression e.

C.6.20.3 Semantics

The value of the powerset expression $\mathbb{P}\,e$ is the set of all subsets of the set that is the value of e.

$$[\![\ \mathbb{P}\,e\ ]\!]^{\mathcal{E}} \quad = \quad \lambda\ M : \mathit{Model} \bullet \mathbb{P}\,([\![\ e\ ]\!]^{\mathcal{E}} M)$$

In terms of the semantic universe, its semantic value, given a model M, is the powerset of values of e in M.

C.6.21 Function and generic operator application

C.6.21.1 Syntax

```
Expression       = ...
                 | Application
                 | ...
                 ;

Application      = PrefixApp
                 | PostfixApp
                 | InfixApp
                 | NofixApp
                 ;

PrefixApp        = PRE , Expression
                 | L , ExpSep , ( Expression , ERE | ExpressionList , SRE ) , Expression
                 ;

PostfixApp       = Expression , POST
                 | Expression , EL , ExpSep , ( Expression , ER | ExpressionList , SR )
                 ;

InfixApp         = Expression , I , Expression
                 | Expression , EL , ExpSep ,
                     ( Expression , ERE | ExpressionList , SRE ) , Expression
                 ;

NofixApp         = L , ExpSep , ( Expression , ER | ExpressionList , SR ) ;
```

C.6.21.2 Transformation

All function operator applications are transformed to annotated application expressions.

All generic operator applications are transformed to annotated generic instantiation expressions.

Each resulting `NAME` should be one for which there is an operator template paragraph in scope (see 12.2.8).

The left-hand sides of many of these transformation rules involve `ExpSep` phrases: they use es metavariables. None of them use ss metavariables: in other words, only the `Expression ES` case of `ExpSep` is specified, not the `ExpressionList SS` case. Where the latter case occurs in a specification, the `ExpressionList` shall be transformed by rule 12.2.12 to an expression, and thence a transformation analogous to that specified for the former case can be performed, differing only in that a ss appears in the function or generic name in place of an es.

C.6.21.3 `PrefixApp`

$$\begin{array}{rcl} pre\ e & \Longrightarrow & pre\bowtie\ e \\ ln\ e_1\ es_1\ ...\ e_{n-2}\ es_{n-2}\ e_{n-1}\ ere\ e_n & \Longrightarrow & ln\bowtie es_1 ... \bowtie es_{n-2} \bowtie ere \bowtie\ (e_1, ..., e_{n-2}, e_{n-1}, e_n) \\ ln\ e_1\ es_1\ ...\ e_{n-2}\ es_{n-2}\ al_{n-1}\ sre\ e_n & \Longrightarrow & ln\bowtie es_1 ... \bowtie es_{n-2} \bowtie sre \bowtie\ (e_1, ..., e_{n-2}, al_{n-1}, e_n) \end{array}$$

$$\begin{array}{rcl} pre\ e & \Longrightarrow & pre\bowtie\ [e] \\ ln\ e_1\ es_1\ ...\ e_{n-2}\ es_{n-2}\ e_{n-1}\ ere\ e_n & \Longrightarrow & ln\bowtie es_1 ... \bowtie es_{n-2} \bowtie ere \bowtie\ [e_1, ..., e_{n-2}, e_{n-1}, e_n] \\ ln\ e_1\ es_1\ ...\ e_{n-2}\ es_{n-2}\ al_{n-1}\ sre\ e_n & \Longrightarrow & ln\bowtie es_1 ... \bowtie es_{n-2} \bowtie sre \bowtie\ [e_1, ..., e_{n-2}, al_{n-1}, e_n] \end{array}$$

C.6.21.4 `PostfixApp`

$$\begin{array}{rcl} e\ post & \Longrightarrow & \bowtie post\ e \\ e_1\ el\ e_2\ es_2\ ...\ e_{n-1}\ es_{n-1}\ e_n\ er & \Longrightarrow & \bowtie el \bowtie es_2 ... \bowtie es_{n-1} \bowtie er\ \ (e_1, e_2, ..., e_{n-1}, e_n) \\ e_1\ el\ e_2\ es_2\ ...\ e_{n-1}\ es_{n-1}\ al_n\ sr & \Longrightarrow & \bowtie el \bowtie es_2 ... \bowtie es_{n-1} \bowtie sr\ \ (e_1, e_2, ..., e_{n-1}, al_n) \end{array}$$

$$\begin{array}{rcl} e\ post & \Longrightarrow & \bowtie post\ [e] \\ e_1\ el\ e_2\ es_2\ ...\ e_{n-1}\ es_{n-1}\ e_n\ er & \Longrightarrow & \bowtie el \bowtie es_2 ... \bowtie es_{n-1} \bowtie er\ [e_1, e_2, ..., e_{n-1}, e_n] \\ e_1\ el\ e_2\ es_2\ ...\ e_{n-1}\ es_{n-1}\ al_n\ sr & \Longrightarrow & \bowtie el \bowtie es_2 ... \bowtie es_{n-1} \bowtie sr\ [e_1, e_2, ..., e_{n-1}, al_n] \end{array}$$

C.6.21.5 `InfixApp`

$$\begin{array}{rcl} e_1\ in\ e_2 & \Longrightarrow & \bowtie in \bowtie\ (e_1, e_2) \\ e_1\ el\ e_2\ es_2\ ...\ e_{n-2}\ es_{n-2}\ e_{n-1}\ ere\ e_n & \Longrightarrow & \bowtie el \bowtie es_2 ... \bowtie es_{n-2} \bowtie ere \bowtie\ (e_1, e_2, ..., e_{n-2}, e_{n-1}, e_n) \\ e_1\ el\ e_2\ es_2\ ...\ e_{n-2}\ es_{n-2}\ al_{n-1}\ sre\ e_n & \Longrightarrow & \bowtie el \bowtie es_2 ... \bowtie es_{n-2} \bowtie sre \bowtie\ (e_1, e_2, ..., e_{n-2}, al_{n-1}, e_n) \end{array}$$

$$\begin{array}{rcl} e_1\ in\ e_2 & \Longrightarrow & \bowtie in \bowtie\ [e_1, e_2] \\ e_1\ el\ e_2\ es_2\ ...\ e_{n-2}\ es_{n-2}\ e_{n-1}\ ere\ e_n & \Longrightarrow & \bowtie el \bowtie es_2 ... \bowtie es_{n-2} \bowtie ere \bowtie\ [e_1, e_2, ..., e_{n-2}, e_{n-1}, e_n] \\ e_1\ el\ e_2\ es_2\ ...\ e_{n-2}\ es_{n-2}\ al_{n-1}\ sre\ e_n & \Longrightarrow & \bowtie el \bowtie es_2 ... \bowtie es_{n-2} \bowtie sre \bowtie\ [e_1, e_2, ..., e_{n-2}, al_{n-1}, e_n] \end{array}$$

C.6.21.6 `NofixApp`

$$\begin{array}{rcl} ln\ e_1\ es_1\ ...\ e_{n-1}\ es_{n-1}\ e_n\ er & \Longrightarrow & ln\bowtie es_1 ... \bowtie es_{n-1} \bowtie er\ \ (e_1, ..., e_{n-1}, e_n) \\ ln\ e_1\ es_1\ ...\ e_{n-1}\ es_{n-1}\ al_n\ sr & \Longrightarrow & ln\bowtie es_1 ... \bowtie es_{n-1} \bowtie sr\ \ (e_1, ..., e_{n-1}, al_n) \end{array}$$

$$ln\ e_1\ es_1\ ...\ e_{n-1}\ es_{n-1}\ e_n\ er \quad \Longrightarrow \quad ln{\bowtie}es_1...{\bowtie}es_{n-1}{\bowtie}er\ [e_1, ..., e_{n-1}, e_n]$$
$$ln\ e_1\ es_1\ ...\ e_{n-1}\ es_{n-1}\ al_n\ sr \quad \Longrightarrow \quad ln{\bowtie}es_1...{\bowtie}es_{n-1}{\bowtie}sr\ [e_1, ..., e_{n-1}, al_n]$$

C.6.21.7 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathbin{⨾} \tau_1}{\Sigma \vdash^{\mathcal{E}} \mathbb{P}(e \mathbin{⨾} \tau_1) \mathbin{⨾} \tau_2}\left(\begin{array}{l}\tau_1 = \mathbb{P}\,\alpha\\ \tau_2 = \mathbb{P}\,\tau_1\end{array}\right)$$

In a powerset expression $\mathbb{P}\,e$, expression e shall be a set. The type of the whole expression is then a powerset of the type of expression e.

C.6.21.8 Semantics

The value of the powerset expression $\mathbb{P}\,e$ is the set of all subsets of the set that is the value of e.

$$[\![\ \mathbb{P}\,e\ ]\!]^{\mathcal{E}} \quad = \quad \lambda\ M : Model \bullet \mathbb{P}\,([\![\ e\ ]\!]^{\mathcal{E}} M)$$

In terms of the semantic universe, its semantic value, given a model M, is the powerset of values of e in M.

C.6.22 Application

C.6.22.1 Syntax

```
Expression        = ...
                  | Expression , Expression
                  | ...
                  ;
```

C.6.22.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \mathbin{⨾} \tau_1 \qquad \Sigma \vdash^{\mathcal{E}} e_2 \mathbin{⨾} \tau_2}{\Sigma \vdash^{\mathcal{E}} (e_1 \mathbin{⨾} \tau_1)\ (e_2 \mathbin{⨾} \tau_2) \mathbin{⨾} \tau_3}\left(\ \tau_1 = \mathbb{P}(\tau_2 \times \tau_3)\ \right)$$

In an application expression $e_1\ e_2$, the expression e_1 shall be a set of pairs, and expression e_2 shall be of the same type as the first components of those pairs. The type of the whole expression is the type of the second components of those pairs.

C.6.22.3 Semantics

The value of the application expression $e_1\ e_2$ is the sole value associated with e_2 in the relation e_1.

$$e_1\ e_2 \mathbin{⨾} \tau \quad \Longrightarrow \quad (\mu\ i : carrier\ \tau \mid (e_2, i) \in e_1 \bullet i)$$

It is semantically equivalent to that sole range value i such that the pair (e_2, i) is in the set of pairs that is the value of e_1. If there is no value or more than one value associated with e_2, then the application expression has a value but what it is is not specified.

C.6.23 Schema decoration

C.6.23.1 Syntax

```
Expression        = ...
                  | Expression , STROKE
                  | ...
                  ;
```


C.6.23.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e \,⨾\, \tau_1}{\Sigma \vdash^{\mathcal{E}} (e \,⨾\, \tau_1)^{\;+} \,⨾\, \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \tau_2 = \mathbb{P}[\{i : dom\ \beta \bullet i\ decor\ ^{+} \mapsto \beta\ i\}] \end{array} \right)$$

In a schema decoration expression $e\ ^{+}$, expression e shall be a schema. The type of the whole expression is that of a schema whose signature is like that of e but with the stroke appended to each of its names.

C.6.23.3 Semantics

The value of the schema decoration expression $e\ ^{+}$ is that schema whose bindings are like those of the schema e except that their names have the addition stroke $^{+}$.

$$(e \,⨾\, \mathbb{P}[i_1 : \tau_1;\ ...;\ i_n : \tau_n])^{+} \quad \Longrightarrow \quad e\ [i_1\ decor\ ^{+} \,/\, i_1, ..., i_n\ decor\ ^{+} \,/\, i_n]$$

It is semantically equivalent to the schema renaming where decorated names rename the original names.

C.6.24 Schema renaming

C.6.24.1 Syntax

```
Expression      = ...
                | Expression , [-tok , DeclName , / , DeclName ,
                      { ,-tok , DeclName , / , DeclName } , ]-tok
                | ...
                ;
```

C.6.24.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e \,⨾\, \tau_1}{\Sigma \vdash^{\mathcal{E}} (e \,⨾\, \tau_1)[j_1 \,/\, i_1, ..., j_n \,/\, i_n] \,⨾\, \tau_2} \left(\begin{array}{l} \#\,\{i_1,\ ...,\ i_n\} = n \\ \tau_1 = \mathbb{P}[\beta_1] \\ \beta_2 = \{j_1 \mapsto i_1,\ ...,\ j_n \mapsto i_n\} ⨾ \beta_1 \cup \{i_1,\ ...,\ i_n\} \mathbin{⩤} \beta_1 \\ \tau_2 = \mathbb{P}[\beta_2] \\ \beta_2 \in (\_ \mathbin{⇸} \_) \end{array} \right)$$

In a schema renaming expression $e\ [j_1 \,/\, i_1, ..., j_n \,/\, i_n]$, there shall be no duplicates amongst the old names $i_1, ..., i_n$. Expression e shall be a schema. The type of the whole expression is that of a schema whose signature is like that of expression e but with the new names in place of corresponding old names. Declarations that are merged by the renaming shall have the same type.

NOTE Old names need not be in the signature of the schema. This is so as to permit renaming to distribute over other notations such as disjunction.

C.6.24.3 Semantics

The value of the schema renaming expression $e\ [j_1 \,/\, i_1, ..., j_n \,/\, i_n]$ is that schema whose bindings are like those of schema e except that some of its names have been replaced by new names, possibly merging components.

$$\begin{array}{rcl} [\![\ e\ [j_1 \,/\, i_1, ..., j_n \,/\, i_n]\ ]\!]^{\mathcal{E}} & = & \lambda\ M : Model \bullet \\ & & \quad \{t_1 : [\![\ e\ ]\!]^{\mathcal{E}} M;\ t_2 : \mathbb{W} \mid \\ & & \qquad t_2 = \{j_1 \mapsto i_1, ..., j_n \mapsto i_n\} ⨾ t_1 \cup \{i_1, ..., i_n\} \mathbin{⩤} t_1 \\ & & \qquad \wedge\ t_2 \in (\_ \mathbin{⇸} \_) \\ & & \quad \bullet\ t_2\} \end{array}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of the bindings (sets of pairs) in the semantic value of e in M with the new names replacing corresponding old names. Where components are merged by the renaming, those components shall have the same value.

C.6.25 Binding selection

C.6.25.1 Syntax

```
Expression        = ...
                  | Expression , . , RefName
                  | ...
                  ;
```

C.6.25.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathbin{⨾} \tau_1}{\Sigma \vdash^{\mathcal{E}} (e \mathbin{⨾} \tau_1) \,.\, i \mathbin{⨾} \tau_2} \left(\begin{array}{l} \tau_1 = [\beta] \\ (i,\ \tau_2) \in \beta \end{array} \right)$$

In a binding selection expression $e\ .\ i$, expression e shall be a binding, and name i shall select one of its components. The type of the whole expression is the type of the selected component.

C.6.25.3 Semantics

The value of the binding selection expression $e\ .\ i$ is that value associated with i in the binding that is the value of e.

$$(e \mathbin{⨾} [\sigma]) \,.\, i \quad\Longrightarrow\quad \{\mathit{carrier}\ [\sigma] \bullet (\mathit{chartuple}\ (\mathit{carrier}\ \ [\sigma]), i)\}\ e$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$(e \mathbin{⨾} [\sigma]) \,.\, i \quad\Longrightarrow\quad (\lambda\ \mathit{carrier}\ [\sigma] \bullet i)\ e$$

It is semantically equivalent to the function construction expression, from bindings of the schema type of e, to the value of the selected name i, applied to the particular binding e.

C.6.26 Tuple selection

C.6.26.1 Syntax

```
Expression        = ...
                  | Expression , . , NUMERAL
                  | ...
                  ;
```

C.6.26.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathbin{⨾} \tau_1}{\Sigma \vdash^{\mathcal{E}} (e \mathbin{⨾} \tau_1) \,.\, b \mathbin{⨾} \tau_2} ((b,\ \tau_2) \in \tau_1)$$

In a tuple selection expression $e\ .\ b$, the type of expression e shall be a Cartesian product, and number b shall select one of its components. The type of the whole expression is the type of the selected component.

C.6.26.3 Semantics

The value of the tuple selection expression $e\ .\ b$ is the b'th component of the tuple that is the value of e.

$$\begin{array}{l} (e \mathbin{⨾} \tau_1 \times ... \times \tau_n) \,.\, b \quad\Longrightarrow\quad \{i : \mathit{carrier}\ (\tau_1 \times ... \times \tau_n) \bullet \\ \qquad\qquad (i, \mu\ i_1 : \mathit{carrier}\ \tau_1;\ ...;\ i_n : \mathit{carrier}\ \tau_n \mid i = (i_1, ..., i_n) \bullet i_b)\}\ e \end{array}$$

NOTE Exploiting notation that is not present in the annotated syntax, this abbreviates to the following.

$$(e \mathbin{⨾} \tau_1 \times ... \times \tau_n) \,.\, b \quad \Longrightarrow \quad (\lambda\ i : carrier\ (\tau_1 \times ... \times \tau_n) \bullet \\ \mu\ i_1 : carrier\ \tau_1;\ ...;\ i_n : carrier\ \tau_n \mid i = (i_1, ..., i_n) \bullet i_b)\ e$$

It is semantically equivalent to the function construction, from tuples of the Cartesian product type to the selected component of the tuple b, applied to the particular tuple e.

C.6.27 Binding construction

C.6.27.1 Syntax

```
Expression      = ...
                | θ , Expression , { STROKE }
                | ...
                ;
```

C.6.27.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e \mathbin{⨾} \tau_1}{\Sigma \vdash^{\mathcal{E}} \theta\,(e \mathbin{⨾} \tau_1)\ ^* \mathbin{⨾} \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \forall\ i : \mathtt{NAME} \mid (i, \alpha_1) \in \beta \bullet (i\ decor\ ^*, \alpha_1) \in \Sigma \land \neg\ \alpha_1 = [\imath_1, ..., \imath_n]\ \alpha_2 \\ \tau_2 = [\beta] \end{array} \right)$$

In a binding construction expression $\theta\, e\ ^*$, the expression e shall be a schema. Every name and type pair in its signature, with the optional decoration added, should be present in the environment with a non-generic type. The type of the whole expression is that of a binding whose signature is that of the schema.

NOTE If the type in the environment were generic, semantic transformation 14.2.5.2 would produce a reference expression whose implicit instantiation is not determined by this International Standard.

C.6.27.3 Semantics

The value of the binding construction expression $\theta\, e\ ^*$ is the binding whose names are those in the signature of schema e and whose values are those of the same names with the optional decoration appended.

$$\theta\, e\ ^* \mathbin{⨾} [i_1 : \tau_1;\ ...;\ i_n : \tau_n] \quad \Longrightarrow \quad \langle\!|\ i_1 == i_1\ decor\ ^*, ..., i_n == i_n\ decor\ ^*\ |\!\rangle$$

It is semantically equivalent to the binding extension expression whose value is that binding.

C.6.28 Reference

C.6.28.1 Syntax

```
Expression      = ...
                | RefName
                | ...
                ;
```

C.6.28.2 Type

In a reference expression, if the name is of the form Δi and no declaration of this name yet appears in the environment, then the following syntactic transformation is applied.

$$\Delta i \quad \stackrel{\Delta i \notin dom\ \Sigma}{\Longrightarrow} \quad [i;\ i\ ']$$

This syntactic transformation makes the otherwise undefined name be equivalent to the corresponding schema construction expression, which is then typechecked.

Similarly, if the name is of the form Ξi and no declaration of this name yet appears in the environment, then the following syntactic transformation is applied.

$$\Xi i \quad \stackrel{\Xi i \notin dom\ \Sigma}{\Longrightarrow} \quad [i;\ i\ ' \mid \theta\, i = \theta\, i\ ']$$

NOTE 1 The Ξ notation is deliberately not defined in terms of the Δ notation.

NOTE 2 Type inference could be done without these syntactic transformations, but they are necessary steps in defining the formal semantics.

NOTE 3 Only occurrences of Δ and Ξ that are in such reference expressions are so transformed, not others such as those in the names of declarations.

$$\frac{}{\Sigma\ \vdash^{\mathcal{E}}\ i\ ⨾\ \tau}\left(\begin{array}{l} i \in dom\ \Sigma \\ \tau = \text{if } \Sigma\ i = [\imath_1, ..., \imath_n]\ \alpha \text{ then } \Sigma\ i, (\Sigma\ i)\ [\alpha_1, ..., \alpha_n] \text{ else } \Sigma\ i \end{array}\right)$$

In any other reference expression i, the name i shall be associated with a type in the environment. If that type is generic, the annotation of the whole expression is a pair of both the uninstantiated type (for the Instantiation clause to determine that this is a reference to a generic definition) and the type instantiated with new distinct variable types (which the context should constrain to non-generic types). Otherwise (if the type in the environment is non-generic), that is the type of the whole expression.

NOTE 4 If the type is generic, the reference expression will be transformed to a generic instantiation expression by the rule in 13.2.3.3. That shall be done only when the implicit instantiations have been determined via constraints on the new variable types $\alpha_1, ..., \alpha_n$.

C.6.28.3 Semantics

The value of a reference expression that refers to a generic definition is an inferred instantiation of that generic definition.

$$i\ ⨾\ [i_1,\ ...,\ i_n]\tau, \tau' \quad \stackrel{\tau' = ([i_1,\ ...,\ i_n]\tau)\ [\alpha_1,\ ...,\ \alpha_n]}{\Longrightarrow} \quad i\ [carrier\ \alpha_1, ..., carrier\ \alpha_n]\ ⨾\ \tau'$$

It is semantically equivalent to the generic instantiation expression whose generic actuals are the carrier sets of the types inferred for the generic parameters. The type τ' is an instantiation of the generic type τ. The types inferred for the generic parameters are α_1, ..., α_n. They shall all be determinable by comparison of τ with τ' as suggested by the condition on the transformation. Cases where these types cannot be so determined, because the generic type is independent of some of the generic parameters, are not well-typed.

EXAMPLE 1 The paragraph

$a[X] == 1$

defines a with type $[X]\mathtt{GIVEN}\ \mathbb{A}$. The paragraph

$b == a$

typechecks, giving the annotated expression $a\ ⨾\ [X]\mathtt{GIVEN}\ \mathbb{A}, \mathtt{GIVEN}\ \mathbb{A}$. Comparison of the generic type with the instantiated type does not determine a type for the generic parameter X, and so this specification is not well-typed.

Cases where these types are not unique (contain unconstrained variables) are not well-typed.

EXAMPLE 2 The paragraph

$empty == \varnothing$

will contain the annotated expression $\varnothing\ ⨾\ [X]\,\mathbb{P}\,X, \mathbb{P}\,\alpha$, in which the type determined for the generic parameter X is unconstrained, and so this specification is not well-typed.

The value of the reference expression that refers to a non-generic definition i is the value of the declaration to which it refers.

$$[\![\ i\ ]\!]^{\mathcal{E}} \;=\; \lambda\ M : Model \bullet M\ i$$

In terms of the semantic universe, its semantic value, given a model M, is that associated with the name i in M.

C.6.29 Generic instantiation

C.6.29.1 Syntax

```
Expression        = ...
                  | RefName , [-tok , Expression , { ,-tok , Expression } , ]-tok
                  | ...
                  ;
```

C.6.29.2 Type

$$\frac{\Sigma\ \vdash^{\mathcal{E}}\ e_1\ ⨾\ \tau_1 \qquad \ldots \qquad \Sigma\ \vdash^{\mathcal{E}}\ e_n\ ⨾\ \tau_n}{\Sigma\ \vdash^{\mathcal{E}}\ i[(e_1\ ⨾\ \tau_1), \ldots, (e_n\ ⨾\ \tau_n)]\ ⨾\ \tau} \left(\begin{array}{l} i \in dom\ \Sigma \\ \Sigma\ i = [\imath_1, \ldots, \imath_n]\ \alpha \\ \tau_1 = \mathbb{P}\,\alpha_1 \\ \quad\vdots \\ \tau_n = \mathbb{P}\,\alpha_n \\ \tau = (\Sigma\ i)\ [\alpha_1, \ldots, \alpha_n] \end{array}\right)$$

In a generic instantiation expression $i\ [e_1, \ldots, e_n]$, the name i shall be associated with a generic type in the environment, and the expressions $e_1, \ldots, e_n$ shall be sets. That generic type shall have the same number of parameters as there are sets. The type of the whole expression is the instantiation of that generic type by the types of those sets' components.

NOTE The operation of generic type instantiation is defined in 13.2.3.1.

C.6.29.3 Semantics

The value of the generic instantiation expression $i\ [e_1, \ldots, e_n]$ is a particular instance of the generic referred to by name i.

$$[\![\ i\ [e_1, \ldots, e_n]\ ]\!]^{\mathcal{E}} \;=\; \lambda\ M : Model \bullet M\ i\ ([\![\ e_1\ ]\!]^{\mathcal{E}} M, \ldots, [\![\ e_n\ ]\!]^{\mathcal{E}} M)$$

In terms of the semantic universe, its semantic value, given a model M, is the generic value associated with the name i in M instantiated with the semantic values of the instantiation expressions in M.

C.6.30 Number literal

C.6.30.1 Introduction

Z accepts the ordinary notation for writing number literals that represent natural numbers, and imposes the usual meaning on those literals. The method of doing this is as follows.

The lexis defines the notion of a numeric string. The prelude defines the notions of natural number, zero, one and addition (of natural numbers). The syntactic transformation rules prescribe how numeric strings are to be understood as natural numbers, using the ideas defined in the prelude.

The extension to integers, and the introduction of other numeric operations on integers, is defined in the mathematical toolkit (annex B).

The extension to other number systems is left to user definition.

C.6.30.2 Syntax

```
Expression        = ...
                  | NUMERAL
                  | ...
                  ;
```

Numeric literals are concrete expressions.

C.6.30.3 Transformation

The value of the multiple-digit number literal expression bc is the number that it denotes.

$$\begin{aligned} bc \quad \Longrightarrow \quad & b + b + b + b + b + \\ & b + b + b + b + b + c \end{aligned}$$

It is semantically equivalent to the sum of ten repetitions of the number literal expression b formed from all but the last digit, added to that last digit.

$$\begin{array}{lll} 0 & \Longrightarrow & number\_literal\_0 \\ 1 & \Longrightarrow & number\_literal\_1 \\ 2 & \Longrightarrow & 1 + 1 \\ 3 & \Longrightarrow & 2 + 1 \\ 4 & \Longrightarrow & 3 + 1 \\ 5 & \Longrightarrow & 4 + 1 \\ 6 & \Longrightarrow & 5 + 1 \\ 7 & \Longrightarrow & 6 + 1 \\ 8 & \Longrightarrow & 7 + 1 \\ 9 & \Longrightarrow & 8 + 1 \end{array}$$

The number literal expressions 0 and 1 are semantically equivalent to *number_literal_0* and *number_literal_1* respectively as defined in section *prelude*. The remaining digits are defined as being successors of their predecessors, using the function + as defined in section *prelude*.

NOTE These syntactic transformations are applied only to `NUMERAL` tokens that form number literal expressions, not to other `NUMERAL` tokens (those in tuple selection expressions and operator template paragraphs), as those other occurrences of `NUMERAL` do not have semantic values associated with them.

C.6.31 Set extension

C.6.31.1 Syntax

```
Expression        = ...
                  | {-tok , [ Expression , { ,-tok , Expression } ] , }-tok
                  | ...
                  ;
```

C.6.31.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \mathbin{⦂} \tau_1 \quad \ldots \quad \Sigma \vdash^{\mathcal{E}} e_n \mathbin{⦂} \tau_n}{\Sigma \vdash^{\mathcal{E}} \{(e_1 \mathbin{⦂} \tau_1), \ldots, (e_n \mathbin{⦂} \tau_n)\} \mathbin{⦂} \tau} \left(\begin{array}{l} \text{if } n > 0 \text{ then} \\ \quad (\tau_1 = \tau_n \\ \quad \vdots \\ \quad \tau_{n-1} = \tau_n \\ \quad \tau = \mathbb{P}\,\tau_1) \\ \text{else } \tau = \mathbb{P}\,\alpha \end{array}\right)$$

In a set extension expression, every component expression shall be of the same type. The type of the whole expression is a powerset of the components' type, or a powerset of a variable type if there are no components. In the latter case, the variable shall be constrained by the context, otherwise the specification is not well-typed.

C.6.31.3 Semantics

The value of the set extension expression $\{ e_1, \ldots, e_n\}$ is the set containing the values of its expressions.

$$[\![\{ e_1, \ldots, e_n\}]\!]^{\mathcal{E}} \;=\; \lambda\, M : \mathit{Model} \bullet \{[\![e_1]\!]^{\mathcal{E}} M, \ \ldots, \ [\![e_n]\!]^{\mathcal{E}} M\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set whose members are the semantic values of the member expressions in M.

C.6.32 Set comprehension

C.6.32.1 Syntax

```
Expression      = ...
                | {-tok , SchemaText , • , Expression , }-tok
                | ...
                ;
```

C.6.32.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \mathbin{⦂} \tau_1 \qquad \Sigma \oplus \beta \vdash^{\mathcal{E}} e_2 \mathbin{⦂} \tau_2}{\Sigma \vdash^{\mathcal{E}} \{(e_1 \mathbin{⦂} \tau_1) \bullet (e_2 \mathbin{⦂} \tau_2)\} \mathbin{⦂} \tau_3} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \tau_3 = \mathbb{P}\,\tau_2 \end{array}\right)$$

In a set comprehension expression $\{e_1 \bullet e_2\}$, expression e_1 shall be a schema. The type of the whole expression is a powerset of the type of expression e_2, as determined in an environment overridden by the signature of schema e_1.

C.6.32.3 Semantics

The value of the set comprehension expression $\{ e_1 \bullet e_2\}$ is the set of values of e_2 for all bindings of the schema e_1.

$$[\![\{ e_1 \bullet e_2\}]\!]^{\mathcal{E}} \;=\; \lambda\, M : \mathit{Model} \bullet \{t_1 : [\![e_1]\!]^{\mathcal{E}} M \bullet [\![e_2]\!]^{\mathcal{E}} \ (M \oplus t_1)\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of values of e_2 in M overridden with a binding value of e_1 in M.

C.6.33 Characteristic set comprehension

C.6.33.1 Syntax

```
Expression      = ...
                | ( ( {-tok , SchemaText , }-tok )  — ( {-tok , }-tok ) )
                                 — ( {-tok , Expression , }-tok )
                | ...
                ;
```

C.6.33.2 Transformation

The value of the characteristic set comprehension expression $\{t\}$ is the set of the values of the characteristic tuple of t.

$$\{t\} \quad \Longrightarrow \quad \{t \bullet \mathit{chartuple}\ t\}$$

It is semantically equivalent to the corresponding set comprehension expression in which the characteristic tuple is made explicit.

C.6.34 Schema construction

C.6.34.1 Syntax

```
Expression      = ...
                | ( [-tok , SchemaText , ]-tok )  — ( [-tok , Expression , ]-tok )
                | ...
                ;
```

C.6.34.2 Transformation

The value of the schema construction expression $[t]$ is that schema whose signature is the names declared by the schema text t, and whose bindings are those that satisfy the constraints in t.

$$[t] \quad \Longrightarrow \quad t$$

It is semantically equivalent to the schema resulting from syntactic transformation of the schema text t.

C.6.34.3 Type

$$\frac{\Sigma\ \vdash^{\mathcal{E}}\ e\ ⨾\ \tau_1}{\Sigma\ \vdash^{\mathcal{E}}\ [i : (e\ ⨾\ \tau_1)]\ ⨾\ \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}\,\alpha \\ \tau_2 = \mathbb{P}[i : \alpha] \end{array} \right)$$

In a variable construction expression $[i : e]$, expression e shall be a set. The type of the whole expression is that of a schema whose signature associates the name i with the type of a member of the set e.

$$\frac{\Sigma\ \vdash^{\mathcal{E}}\ e\ ⨾\ \tau_1 \qquad \Sigma \oplus \beta\ \vdash^{\mathcal{P}}\ p}{\Sigma\ \vdash^{\mathcal{E}}\ [(e\ ⨾\ \tau_1) \mid p]\ ⨾\ \tau_2} \left(\begin{array}{l} \tau_1 = \mathbb{P}[\beta] \\ \tau_2 = \tau_1 \end{array} \right)$$

In a schema construction expression $[e \mid p]$, expression e shall be a schema, and predicate p shall be well-typed in an environment overridden by the signature of schema e. The type of the whole expression is the same as the type of expression e.

C.6.34.4 Semantics

The value of the variable construction expression $[i : e]$ is the set of all bindings whose sole name is i and whose associated value is in the set that is the value of e.

$$[\![\ [i : e]\ ]\!]^{\mathcal{E}} \quad = \quad \lambda\ M : \mathit{Model} \bullet \{w : [\![\ e\ ]\!]^{\mathcal{E}} M \bullet \{i \mapsto w\}\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of all singleton bindings (sets of pairs) of the name i associated with a value from the set that is the semantic value of e in M.

The value of the schema construction expression $[e \mid p]$ is the set of all bindings of schema e that satisfy the constraints of predicate p.

$$[\![\, [e \mid p] \,]\!]^{\mathcal{E}} \;=\; \lambda\, M : \mathit{Model} \bullet \{t : [\![\, e \,]\!]^{\mathcal{E}} M \mid M \oplus t \in [\![\, p \,]\!]^{\mathcal{P}} \bullet t\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of the bindings (sets of pairs) that are members of the semantic value of schema e in M such that p is *true* in the model M overridden with that binding.

C.6.35 Binding extension

C.6.35.1 Syntax

```
Expression        = ...
                  | ⦉ , [ DeclName , == , Expression ,
                        { ,-tok , DeclName , == , Expression } ] , ⦊
                  | ...
                  ;
```

C.6.35.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \,⦂\, \tau_1 \quad \ldots \quad \Sigma \vdash^{\mathcal{E}} e_n \,⦂\, \tau_n}{\Sigma \vdash^{\mathcal{E}} \langle\!|\, i_1 == (e_1 \,⦂\, \tau_1), \ldots, i_n == (e_n \,⦂\, \tau_n) \,|\!\rangle \,⦂\, \tau} \begin{pmatrix} \#\,\{i_1, \ldots, i_n\} = n \\ \tau = [i_1 : \tau_1; \ldots; i_n : \tau_n] \end{pmatrix}$$

In a binding extension expression $\langle\!|\, i_1 == e_1, \ldots, i_n == e_n \,|\!\rangle$, there shall be no duplication amongst the bound names. The type of the whole expression is that of a binding whose signature associates the names with the types of the corresponding expressions.

C.6.35.3 Semantics

The value of the binding extension expression $\langle\!|\, i_1 == e_1, \ldots, i_n == e_n \,|\!\rangle$ is the binding whose names are as enumerated and whose values are those of the associated expressions.

$$[\![\, \langle\!|\, i_1 == e_1, \ldots, i_n == e_n \,|\!\rangle \,]\!]^{\mathcal{E}} \;=\; \lambda\, M : \mathit{Model} \bullet \{i_1 \mapsto [\![\, e_1 \,]\!]^{\mathcal{E}} M, \ldots, i_n \mapsto [\![\, e_n \,]\!]^{\mathcal{E}} M\}$$

In terms of the semantic universe, its semantic value, given a model M, is the set of pairs enumerated by its names each associated with the semantic value of the associated expression in M.

C.6.36 Tuple extension

C.6.36.1 Syntax

```
Expression        = ...
                  | (-tok , Expression , ,-tok , Expression , { ,-tok , Expression } , )-tok
                  | ...
                  ;
```

C.6.36.2 Type

$$\frac{\Sigma \vdash^{\mathcal{E}} e_1 \,⦂\, \tau_1 \quad \ldots \quad \Sigma \vdash^{\mathcal{E}} e_n \,⦂\, \tau_n}{\Sigma \vdash^{\mathcal{E}} ((e_1 \,⦂\, \tau_1), \ldots, (e_n \,⦂\, \tau_n)) \,⦂\, \tau} \left(\tau = \tau_1 \times \ldots \times \tau_n \right)$$

In a tuple extension expression $(e_1, ..., e_n)$, the type of the whole expression is the Cartesian product of the types of the individual component expressions.

C.6.36.3 Semantics

The value of the tuple extension expression $(e_1, ..., e_n)$ is the tuple containing the values of its expressions in order.

$$[\![(e_1, ..., e_n)]\!]^{\mathcal{E}} \quad = \quad \lambda\ M : \mathit{Model} \bullet ([\![e_1]\!]^{\mathcal{E}} M,\ ...,\ [\![e_n]\!]^{\mathcal{E}} M)$$

In terms of the semantic universe, its semantic value, given a model M, is the tuple whose components are the semantic values of the component expressions in M.

C.6.37 Characteristic definite description

C.6.37.1 Syntax

```
Expression      = ...
                | (-tok , μ , SchemaText , )-tok
                | ...
                ;
```

C.6.37.2 Transformation

The value of the characteristic definite description expression $(\mu\ t)$ is the sole value of the characteristic tuple of schema text t.

$$(\mu\ t) \quad \Longrightarrow \quad \mu\ t \bullet \mathit{chartuple}\ t$$

It is semantically equivalent to the corresponding definite description expression in which the characteristic tuple is made explicit.

C.6.38 Parenthesized expression

C.6.38.1 Syntax

```
Expression      = ...
                | (-tok , Expression , )-tok
                | ...
                ;
```

C.6.38.2 Transformation

The value of the parenthesized expression (e) is the value of expression e.

$$(e) \quad \Longrightarrow \quad e$$

It is semantically equivalent to e.

C.7 Schema text

C.7.1 Introduction

A `SchemaText` introduces local variables, with constraints on their values.

C.7.2 Syntax

```
SchemaText     = [ DeclPart ] , [ |-tok , Predicate ] ;

DeclPart       = Declaration , { ( ;-tok | NL ) , Declaration } ;

Declaration    = DeclName , { ,-tok , DeclName } , : , Expression
               | DeclName , == , Expression
               | Expression
               ;
```

C.7.3 Transformation

There is no separate schema text class in the annotated syntax: all concrete schema texts are transformed to expressions.

C.7.3.1 `Declaration`

Each declaration is transformed to an equivalent expression.

A constant declaration is equivalent to a variable declaration in which the variable ranges over a singleton set.

$$i == e \quad \Longrightarrow \quad i : \{e\}$$

A comma-separated multiple declaration is equivalent to the conjunction of variable construction expressions in which all variables are constrained to be of the same type.

$$i_1, ..., i_n : e \quad \Longrightarrow \quad [i_1 : e \ ⨾ \ \tau_1] \land ... \land [i_n : e \ ⨾ \ \tau_1]$$

C.7.3.2 `DeclPart`

Each declaration part is transformed to an equivalent expression.

$$de_1;\ ...;\ de_n \quad \Longrightarrow \quad de_1 \land ... \land de_n$$

If `NL` tokens have been used in place of any ; s, the same transformation to $\land$ applies.

C.7.3.3 `SchemaText`

Given the above transformations of `Declaration` and `DeclPart`, any `DeclPart` in a `SchemaText` can be assumed to be a single expression.

A `SchemaText` with non-empty `DeclPart` and `Predicate` is equivalent to the schema construction expression containing that schema text.

$$e \mid p \quad \Longrightarrow \quad [e \mid p]$$

If both `DeclPart` and `Predicate` are omitted, the schema text is equivalent to the set containing the empty binding.

$$\Longrightarrow \quad \{\langle\!\mid\ \mid\!\rangle\}$$

If just the `DeclPart` is omitted, the schema text is equivalent to the schema construction expression in which there is a set containing the empty binding.

$$\mid p \quad \Longrightarrow \quad [\{\langle\!\mid\ \mid\!\rangle\} \mid p]$$

Annex D
(informative)
Tutorial

D.1 Introduction

The aim of this tutorial is to show, by examples, how this International Standard can be used to determine whether a specification is a well-formed Z sentence, and if it is, to determine its semantics. The examples cover some of the more difficult parts of Z, and some of the recent innovations in the Z notation.

D.2 Semantics as models

The semantics of a specification is determined by sets of models, each model being a function from names defined by the specification to values that those names are permitted to have by the constraints imposed on them in the specification. For example, consider the following specification.

$$\begin{array}{|l} n : \mathbb{N} \\ \hline n \in \{1, 2, 3, 4\} \end{array}$$

This specification introduces one name with four possible values (ignoring the prelude section for the moment). The set of models defining the meaning of the specification contains four models, as follows.

$$\{\{n \mapsto 1\}, \{n \mapsto 2\}, \{n \mapsto 3\}, \{n \mapsto 4\}\}$$

One model of the prelude section can be written as follows.

$$\begin{array}{l} \{\mathbb{A} \mapsto \mathbb{A}, \\ \mathbb{N} \mapsto \mathbb{N}, \\ number\_literal\_0 \mapsto 0, \\ number\_literal\_1 \mapsto 1, \\ \_ + \_ \mapsto \{((0,0),0), ((0,1),1), ((1,0),1), ((1,1),2), ...\}\} \end{array}$$

The behaviour of $(\_ + \_)$ on non-natural numbers, e.g. reals, has not been defined at this point, so the set of models for the prelude section includes alternatives for every possible extended behaviour of addition. The set of models for the whole specification arises from extending models of the prelude section with associations for n in all combinations.

This International Standard specifies the relation between Z specifications and their semantics in terms of sets of models. That relation is specified as a composition of relations, each implementing a phase within the standard. Those phases are as identified in Figure 1 in the conformance clause, namely mark-up, lexing, parsing, characterising, syntactic transformation, type inference, semantic transformation and semantic relation. The rest of this tutorial illustrates the effects of those phases on example Z phrases.

D.3 Given types and schema definition paragraphs

The following two paragraphs are taken from the birthday book specification [12].

$$[NAME, DATE]$$

$$\begin{array}{|l} \multicolumn{1}{l}{BirthdayBook} \\ \hline known : \mathbb{P}\, NAME \\ birthday : NAME \pfun DATE \\ \hline known = dom\ birthday \\ \hline \end{array}$$

The mark-up, lexing, parsing and syntactic transformation phases are illustrated using this example.

D.3.1 Mark-ups

The mathematical representation of Z is what one would write with pen, pencil, chalk, etc. Instructing a computer to produce the same appearance currently requires the use of a mark-up language. There are many different mark-up languages, each tailored to different circumstances, such as particular typesetting software. This International Standard defines some mark-ups in annex A, by relating substrings of the mark-up language to strings of Z characters. Source text for the birthday-book paragraphs written in the mark-ups defined in annex A follow. The translation of these into sequences of Z characters is not explained here – annex A provides sufficient information.

D.3.1.1 LaTeX mark-up

```
\begin{zed}
[NAME, DATE]
\end{zed}

\begin{schema}{BirthdayBook}
known : \power NAME\\
birthday : NAME \pfun DATE
\where
known = \dom~birthday
\end{schema}
```

D.3.1.2 Email mark-up

```
[NAME, DATE]

+-- BirthdayBook ---
known : %P NAME
birthday : NAME -+-> DATE
|--
known = dom birthday
---
```

D.3.2 Lexing

Lexing is the translation of a specification's sequence of Z characters to a corresponding sequence of tokens. The translation is defined by the lexis in clause 7. Associated with the tokens `NAME` and `NUMERAL` are the original names and numerals. Here on the left is the sequence of tokens corresponding to the extract from the birthday book (with `-tok` suffices omitted), and on the right is the same sequence but revealing the underlying spelling of the name tokens.

$[\mathtt{NAME}, \mathtt{NAME}]$ `END`	$[NAME, DATE]$ `END`
`SCH NAME`	`SCH` $BirthdayBook$
`NAME` : $\mathbb{P}$ `NAME NL`	$known : \mathbb{P}\, NAME$ `NL`
`NAME` : `NAME I NAME`	$birthday : NAME \pfun DATE$
$\mid$	$\mid$
`NAME` $=$ `NAME NAME`	$known = dom\ birthday$
`END`	`END`

The layout here is of no significance: there are `NL` and `END` tokens where ones are needed. The paragraph outline has been replaced by a `SCH` box token, to satisfy the linear syntax requirement of the syntactic metalanguage. `NAME` and `I` are name tokens: this abstraction allows the fixed size grammar of the concrete syntax to cope with the extensible Z notation.

This specification's sequence of Z characters does conform to the lexis. If it had not, then subsequent phases would not be applicable, and this International Standard would not define a meaning for the specification.

D.3.3 Parsing

Parsing is the translation of the sequence of tokens produced by lexing to a tree structure, grouping the tokens into grammatical phrases. The grammar is defined by the concrete syntax in clause 8. The parse tree for the birthday-book specification is shown in Figure D.1.

The sequence of tokens produced by lexing can be seen in this tree by reading just the leaf nodes in order from left to right. To save space elsewhere in this International Standard, parse trees are presented as just their textual fringes, with parentheses added where necessary to override precedences.

This specification's sequence of tokens does conform to the concrete syntax. If it had not, then subsequent phases would not be applicable, and this International Standard would not define a meaning for the specification.

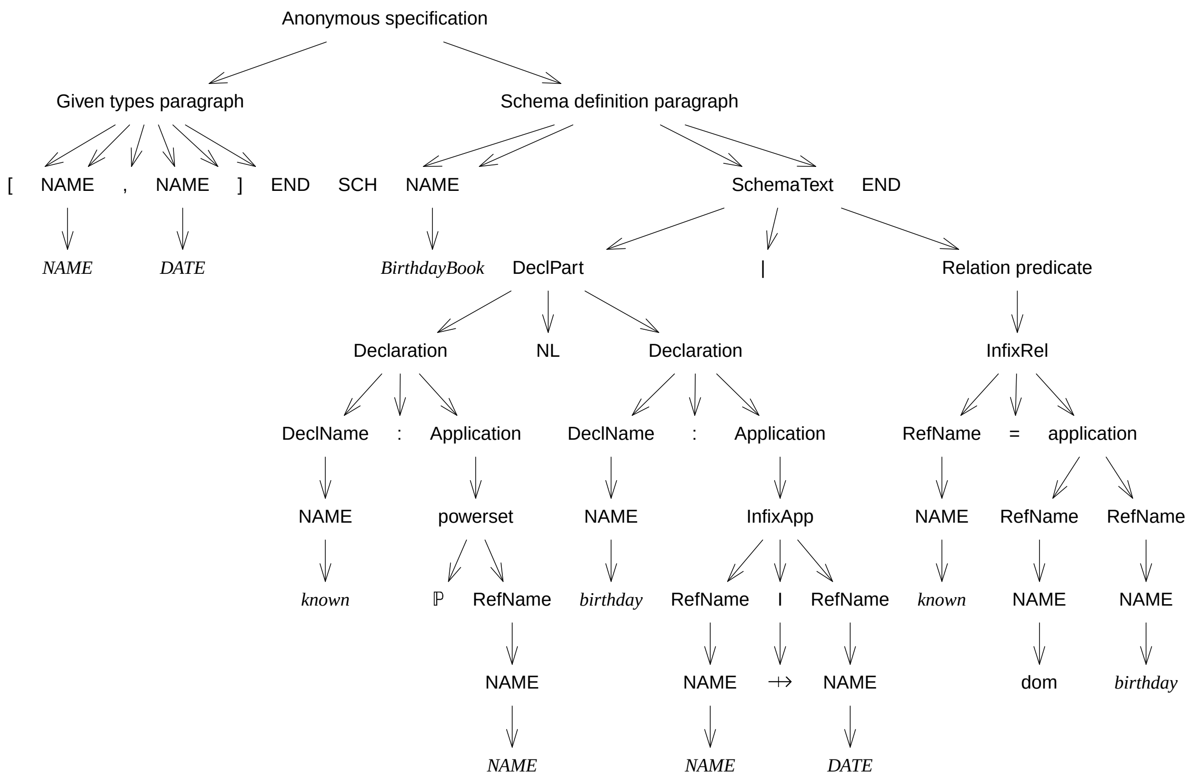
D.3.4 Syntactic transformation

The meaning of a Z specification is established by relating it to an interpretation in a semantic universe. That relation is expressed using ZF set theory, which is not itself formally defined. It is therefore beneficial to define as much Z notation as possible by transformations to other Z notation, so that only a relatively small kernel need be related using ZF set theory. Conveniently, that Z kernel contains largely notation that has direct counterparts in traditional ZF set theory, the novel Z notation having been largely transformed away. A further benefit is that the transformations reveal relationships between different Z notations. The syntactic transformation stage is one of several phases of such transformation.

The syntactic transformation rules (clause 12) are applied to a parsed sentence of the concrete syntax (clause 8). The notation that results is a sentence of the annotated syntax (clause 10).

Consider the effect of the syntactic transformation rules on the birthday book extract. There is no syntactic

Figure D.1 – Parse tree of birthday book example



transformation rule for given types; given types are in the annotated syntax. So the first paragraph is left unchanged.

[NAME,NAME] END	[*NAME*,*DATE*] END

The schema paragraph requires several syntactic transformations before it becomes a sentence of the annotated syntax. The order in which these syntactic transformations are applied does not matter, as the same result is obtained.

Transform `NL` by first rule in 12.2.7.

SCH NAME	SCH *BirthdayBook*
NAME : ℙ NAME;	*known* : ℙ *NAME*;
NAME : NAME I NAME	*birthday* : *NAME* ⇸ *DATE*
|	|
NAME = NAME NAME	*known* = *dom birthday*
END	END

Transform generic application *NAME* ⇸ *DATE* by sixth `InfixApp` rule in 12.2.11.

SCH NAME	SCH *BirthdayBook*
NAME : ℙ NAME;	*known* : ℙ *NAME*;
NAME : NAME [NAME, NAME]	*birthday* : ⋈⇸⋈ [*NAME*, *DATE*]
|	|
NAME = NAME NAME	*known* = *dom birthday*
END	END

Transform equality by third `InfixRel` rule in 12.2.10.

SCH NAME	SCH *BirthdayBook*
NAME : ℙ NAME;	*known* : ℙ *NAME*;
NAME : NAME [NAME, NAME]	*birthday* : ⋈⇸⋈ [*NAME*, *DATE*]
|	|
NAME ∈ {NAME NAME}	*known* ∈ {*dom birthday*}
END	END

Transform basicdecls by sixth rule in 12.2.7.

SCH NAME	SCH *BirthdayBook*
[NAME : ℙ NAME];	[*known* : ℙ *NAME*];
[NAME : NAME [NAME, NAME]]	[*birthday* : ⋈⇸⋈ [*NAME*, *DATE*]]
|	|
NAME ∈ {NAME NAME}	*known* ∈ {*dom birthday*}
END	END

Transform schema text by second rule in 12.2.7.

SCH NAME	SCH *BirthdayBook*
[[NAME : ℙ NAME] ∧	[[*known* : ℙ *NAME*] ∧
[NAME : NAME [NAME, NAME]]	[*birthday* : ⋈⇸⋈ [*NAME*, *DATE*]]
|	|
NAME ∈ {NAME NAME}]	*known* ∈ {*dom birthday*}]
END	END

Transform paragraph by first rule in 12.2.3.

AX	AX
[NAME ==	$[BirthdayBook ==$
[[NAME : $\mathbb{P}$ NAME] $\wedge$	$[[known : \mathbb{P}\, NAME] \wedge$
[NAME : NAME [NAME, NAME]]	$[birthday : \bowtie\!\!\leftrightarrow\!\!\bowtie\, [NAME, DATE]]$
|	|
NAME $\in$ {NAME NAME}]]	$known \in \{dom\ birthday\}]$
END	END

The two paragraphs now form a sentence of the annotated syntax. These syntactic transformations do not change the meaning: the meaning of the annotated representation is the same as that of the original schema paragraph. This is ensured, despite transformations to notations of different precedences, by transforming trees not text – the trees are presented as text above solely to save space.

Do not be surprised that the result of syntactic transformation looks "more complicated" than the original formulation – if it did not, there would not have been much point in having the notation that has been transformed away. The benefits are that fewer notations remain to be defined, and those that have been defined have been defined entirely within Z.

D.4 Axiomatic description paragraphs

Here is a very simple axiomatic description paragraph, preceded by an auxiliary given types paragraph.

$[X]$

| $i : X$

D.4.1 Lexing and parsing

Lexing generates the following sequence of tokens, with corresponding spellings of name tokens.

[NAME] END	$[X]$ END
AX NAME : NAME END	AX $i : X$ END

Parsing proceeds as for the birthday book, so is not explained in detail again.

D.4.2 Syntactic transformation

The schema text is transformed to an expression.

AX [NAME : NAME] END	AX $[i : X]$ END

This is now a sentence of the annotated syntax.

D.4.3 Type inference

The type inference phase adds annotations to each expression and each paragraph in the parse tree. Without going into too much detail, the signature of the given types paragraph is determined by rule 13.2.4.1 to be $X : \mathbb{P}(\mathtt{GIVEN}\ X)$.

$[X]$ END $⨾\ X : \mathbb{P}(\mathtt{GIVEN}\ X)$

Rule 13.2.2.1 adds the name of the given type X to the type environment, associated with type $\mathbb{P}(\mathtt{GIVEN}\ X)$. The annotation for the reference expression referring to X is determined by rule 13.2.6.1 using that type environment to be $\mathbb{P}(\mathtt{GIVEN}\ X)$. Hence the type of the variable construction expression is found by rule 13.2.6.13 to be $\mathbb{P}[i : \mathtt{GIVEN}\ X]$. Hence the signature of the axiomatic description paragraph is determined. The resulting annotated tree is shown in Figure D.2, and as linear text as follows.

AX $([i : (X\ ⨾\ \mathbb{P}(\mathtt{GIVEN}\ X))]\ ⨾\ \mathbb{P}[i : \mathtt{GIVEN}\ X])$ END $⨾\ i : \mathtt{GIVEN}\ X$

This specification's parse tree is well-typed. If it were not, then subsequent phases would not be applicable, and this International Standard would not define a meaning for the specification.

D.4.4 Semantic relation

The semantic relation phase takes a sentence of the annotated syntax and relates it to its meaning in terms of sets of models. The meaning of a paragraph d is given by the semantic relation $[\![\ d\ ]\!]^{\mathcal{D}}$, which relates a model to that same model extended with associations between its names and their semantic values in the given model. For the example given types paragraph, the semantic relation in 15.2.3.1 relates any model to that model extended with an association of the given type name X with an arbitrarily chosen set w. (A further association is made between a distinctly decorated version of the given type name $X\heartsuit$ and that same semantic value, for use in avoiding variable capture.)

$\{X \mapsto w, X\heartsuit \mapsto w\}$

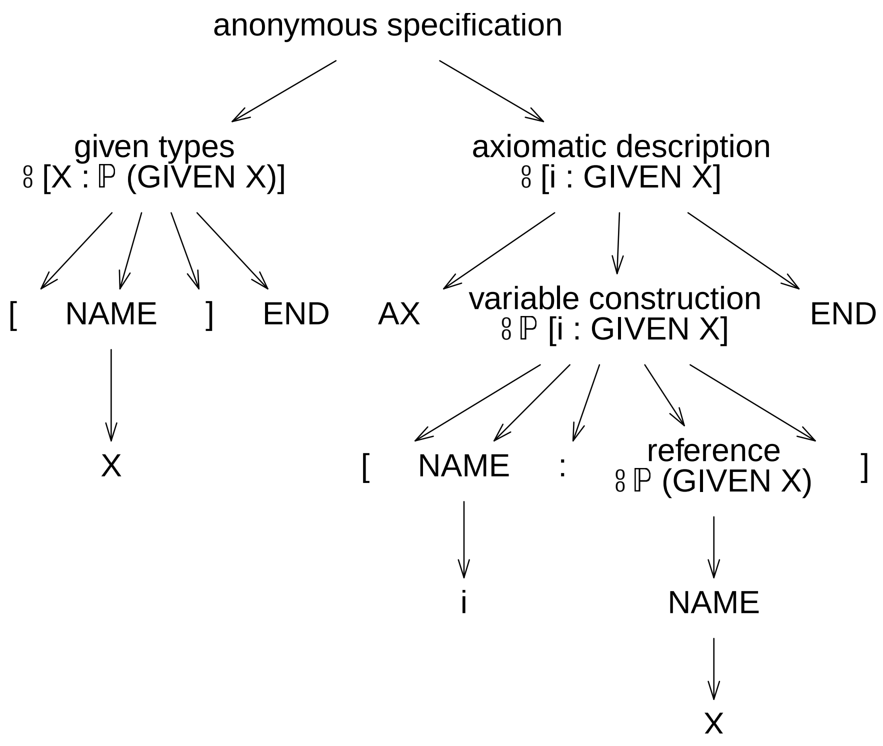
This is one model of the prefix of the specification up to the given types paragraph (ignoring the prelude). The set of models defining the meaning of this prefix includes other models, each with a different set w.

The meaning of an expression e is given by the semantic function $[\![\ e\ ]\!]^{\mathcal{E}}$, which maps the expression to its semantic value in a given model. Within the axiomatic description paragraph, the reference to the given type X has a semantic value determined by relation 15.2.5.1 as being the semantic value w already associated with the given type name X in the model. The variable construction expression $[i : X]$ has a semantic value determined by relation 15.2.5.9 that represents a set of bindings of the name i to members of the semantic value of the reference to X, i.e. the set w. The meaning of the example axiomatic description paragraph, as given by semantic relation 15.2.3.2, is to relate any model to that model extended with a binding that is in the set that is the semantic value of the variable construction expression. So, if the members of w are $w_1, w_2, ...$, then the set of models defining the meaning of the specification includes the following.

$\{\{X \mapsto w, X\heartsuit \mapsto w, i \mapsto w_1\}, \{X \mapsto w, X\heartsuit \mapsto w, i \mapsto w_2\}, ...\}$

If w is chosen to be the empty set, then the set of models is empty.

Figure D.2 – Annotated parse tree of part of axiomatic example



D.5 Generic axiomatic description paragraphs

Here is a generic axiomatic description paragraph. Although it looks simple, it has the complication of being a loose generic definition.

$$\begin{array}{|l} [X] \\ \hline i : \mathbb{P}\, X \end{array}$$

D.5.1 Lexing and parsing

Lexing generates the following sequence of tokens, with corresponding spellings of name tokens.

$\mathtt{GENAX}\ [\mathtt{NAME}]\ \mathtt{NAME} : \mathbb{P}\,\mathtt{NAME}\ \mathtt{END}$ $\qquad$ $\mathtt{GENAX}\ [X]\ i : \mathbb{P}\, X\ \mathtt{END}$

Parsing proceeds as for the birthday book, so is not explained in detail again.

D.5.2 Syntactic transformation

The schema text is transformed to an expression.

$\mathtt{GENAX}\ [\mathtt{NAME}]\ [\mathtt{NAME} : \mathbb{P}\,\mathtt{NAME}]\ \mathtt{END}$ $\qquad$ $\mathtt{GENAX}\ [X]\ [i : \mathbb{P}\, X]\ \mathtt{END}$

This is now a sentence of the annotated syntax.

D.5.3 Type inference

Without going into too much detail, rule 13.2.4.3 adds the name of the generic parameter X to the type environment, associated with type $\mathbb{P}(\mathtt{GENTYPE}\ X)$. The annotation for the reference expression referring to X is determined by rule 13.2.6.1 using that type environment to be $\mathbb{P}(\mathtt{GENTYPE}\ X)$. Hence the type of the powerset expression is found by rule 13.2.6.5 to be $\mathbb{P}\,\mathbb{P}(\mathtt{GENTYPE}\ X)$. Hence the type of the variable construction expression is found by rule 13.2.6.13 to be $\mathbb{P}[i : \mathbb{P}(\mathtt{GENTYPE}\ X)]$. Hence the signature of the generic axiomatic description paragraph is determined.

$\mathtt{GENAX}\ [X]\ ([i : (\mathbb{P}(X \mathbin{⨾} \mathbb{P}(\mathtt{GENTYPE}\ X)) \mathbin{⨾} \mathbb{P}\,\mathbb{P}(\mathtt{GENTYPE}\ X))] \mathbin{⨾} \mathbb{P}[i : \mathbb{P}(\mathtt{GENTYPE}\ X)])\ \mathtt{END} \mathbin{⨾} i : \mathbb{P}(\mathtt{GENTYPE}\ X)$

D.5.4 Semantic relation

Every use of a generic definition instantiates the generic parameters with particular sets. A suitable semantic value for a generic definition is therefore a function from the semantic values of the sets instantiating the generic parameters to the semantic value of the definition given those values for the parameters. In the case of a loose generic definition, several models are needed to express its semantics, each giving a different function. Each model defining the meaning of the example generic axiomatic description paragraph has the following form.

$\{i \mapsto \{s_1 \mapsto \mathit{subset\ of}\ s_1,\ s_2 \mapsto \mathit{subset\ of}\ s_2,\ ...\}\}$

The model associates the name i with a function from sets $s_1, s_2, ...$ instantiating the generic parameter X to the semantic value of i resulting from the corresponding instantiation. Different models for this paragraph have different subsets of $s_1, s_2, ...$ The way this is defined by the semantic relations is roughly as follows.

The semantic relation in 15.2.3.3 specifies that w is a binding in the semantic value of schema e in the given model extended with an association of the generic parameter X with the set instantiating it w_1. (The model is also extended with a further association between a distinctly decorated version of the generic parameter $X\spadesuit$ and that same semantic value w_1, for use in avoiding variable capture). This is specified in a way that is cautious of e being undefined in the extended model. The determination of the value of e is done in the same way as for the preceding example, using semantic relations 15.2.5.5, 15.2.5.1 and 15.2.5.9. The semantic relation for the paragraph is then able to extend its given model with the association illustrated above.

D.6 Operator templates and generics

The definition of relations in the toolkit provides an example of an operator template and the definition of a generic operator.

generic 5 rightassoc $(\_ \leftrightarrow \_)$

$| \quad X \leftrightarrow Y == \mathbb{P}(X \times Y)$

D.6.1 Lexing and parsing

An operator template paragraph affects the lexing and parsing of subsequent paragraphs. In this example, it causes subsequent appearances of names using the word $\leftrightarrow$ to be lexed as I tokens, and hence its infix applications are parsed as operator names (illustrated in the example) or as generic operator application expressions. An operator template paragraph does not have any further effect on the meaning of a specification, so a parsed representation is needed of only the generic operator definition paragraph, for which lexing generates the following sequence of tokens and corresponding spellings of name tokens.

$\mathtt{NAME}\ \mathtt{I}\ \mathtt{NAME} == \mathbb{P}(\mathtt{NAME} \times \mathtt{NAME})\ \mathtt{END}$	$X \leftrightarrow Y == \mathbb{P}(X \times Y)\ \mathtt{END}$

Parsing proceeds as for the birthday-book example, so is not explained in detail again.

D.6.2 Syntactic transformation

The generic operator name is transformed by the first InfixGenName rule in 12.2.9.

$\_\ \mathtt{I}\ \_\ [\mathtt{NAME}, \mathtt{NAME}] == \mathbb{P}(\mathtt{NAME} \times \mathtt{NAME})\ \mathtt{END}$	$\_ \leftrightarrow \_\ [X, Y] == \mathbb{P}(X \times Y)\ \mathtt{END}$

This generic horizontal definition paragraph is then transformed by 12.2.3.4 to a generic axiomatic description paragraph, which is the sole form of generic definition for which there is a semantic relation.

$\mathtt{GENAX}\ [\mathtt{NAME}, \mathtt{NAME}]$ $\_\ \mathtt{I}\ \_ == \mathbb{P}(\mathtt{NAME} \times \mathtt{NAME})$ $\mathtt{END}$	$\mathtt{GENAX}\ [X, Y]$ $\_ \leftrightarrow \_ == \mathbb{P}(X \times Y)$ $\mathtt{END}$

Transform Cartesian product expression by the rule in 12.2.6.8 to a set of pairs.

$\mathtt{GENAX}\ [\mathtt{NAME}, \mathtt{NAME}]$ $\_\ \mathtt{I}\ \_ == \mathbb{P}\{\mathtt{NAME} : \mathtt{NAME};\ \mathtt{NAME} : \mathtt{NAME} \bullet (\mathtt{NAME}, \mathtt{NAME})\}$ $\mathtt{END}$	$\mathtt{GENAX}\ [X, Y]$ $\_ \leftrightarrow \_ == \mathbb{P}\{x : X;\ y : Y \bullet (x, y)\}$ $\mathtt{END}$

Transform the operator name by the first InfixName rule in 12.2.8.3.

$\mathtt{GENAX}\ [\mathtt{NAME}, \mathtt{NAME}]$ $\mathtt{NAME} == \mathbb{P}\{\mathtt{NAME} : \mathtt{NAME};\ \mathtt{NAME} : \mathtt{NAME} \bullet (\mathtt{NAME}, \mathtt{NAME})\}$ $\mathtt{END}$	$\mathtt{GENAX}\ [X, Y]$ $\bowtie\!\leftrightarrow\!\bowtie == \mathbb{P}\{x : X;\ y : Y \bullet (x, y)\}$ $\mathtt{END}$

Transform the schema texts to expressions.

$\mathtt{GENAX}\ [\mathtt{NAME}, \mathtt{NAME}]$ $[\mathtt{NAME} : \{\mathbb{P}\{[\mathtt{NAME} : \mathtt{NAME}] \wedge [\mathtt{NAME} : \mathtt{NAME}] \bullet (\mathtt{NAME}, \mathtt{NAME})\}\}]$ $\mathtt{END}$	$\mathtt{GENAX}\ [X, Y]$ $[\bowtie\!\leftrightarrow\!\bowtie : \{\mathbb{P}\{[x : X] \wedge [y : Y] \bullet (x, y)\}\}]$ $\mathtt{END}$

This is now a sentence of the annotated syntax.

D.6.3 Type inference

The type inference phase adds annotations to the parse tree. For this example, the resulting subtree for the set extension expression (to the right of the colon) is shown in Figure D.3.

Informally, the way the annotations are determined is as follows. The type inference rule for generic axiomatic description paragraph (13.2.4.3) overrides the type environment with types for the generic parameters X and Y. The type inference rule for reference expression (13.2.6.1) retrieves these types for the references to X and Y. The type inference rule for variable construction expression (13.2.6.13) builds the schema types. The type inference rule for schema conjunction expression (13.2.6.16) merges those schema types. The type inference rule for set comprehension expression (13.2.6.4) overrides the type environment with types for the local declarations

x and y. The type inference rule for reference expression (13.2.6.1) retrieves these types for the references to x and y. The type inference rule for tuple extension expression (13.2.6.6) builds the Cartesian product type. The type of the set comprehension is thus determined, and hence that of the powerset by rule 13.2.6.5 and that of the set extension by rule 13.2.6.3.

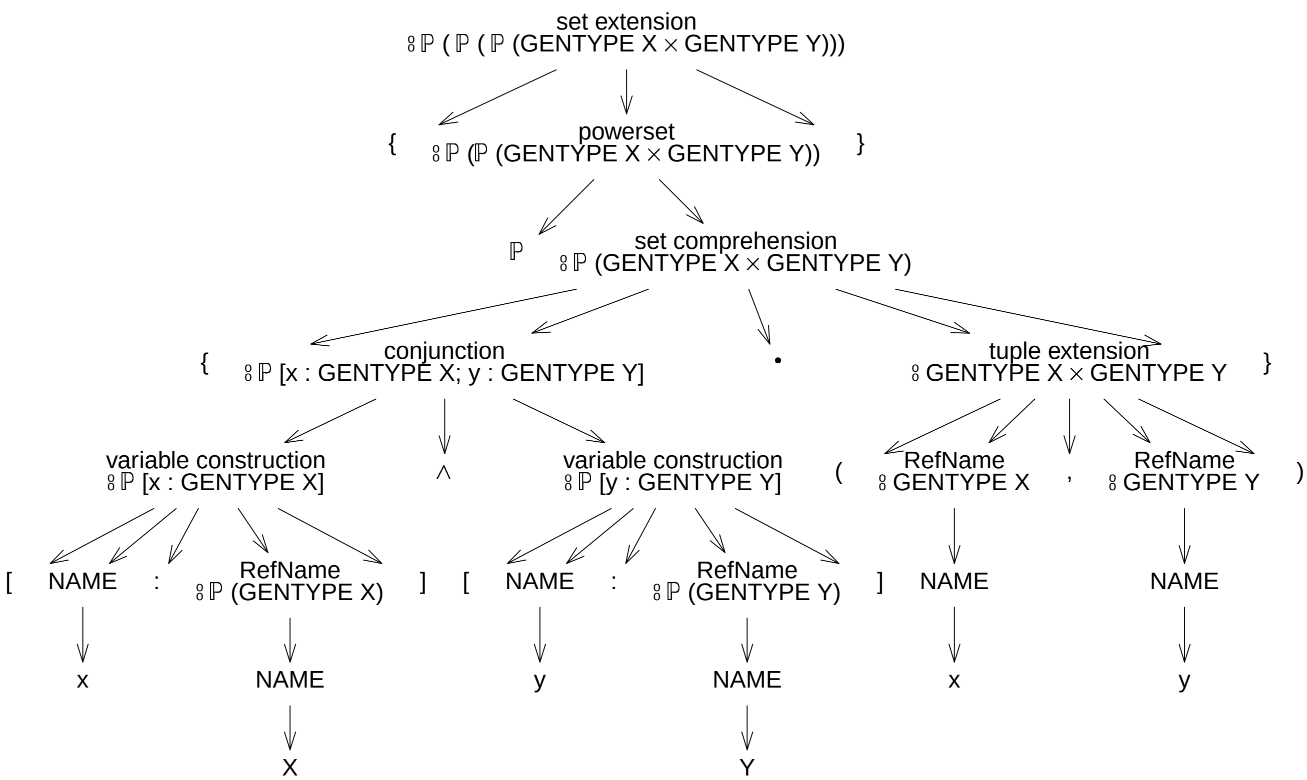
D.6.4 Semantic relation

For the example generic axiomatic description paragraph, the semantic relation in 15.2.3.3 associates name $\_ \leftrightarrow \_$ with a function from the semantic values of the sets instantiating the generic parameters X and Y to the semantic value of the powerset expression given those values for X and Y. The semantic value of the example's powerset expression is given by semantic relations 15.2.5.4, 15.2.5.5 and 15.2.5.6 as sets of tuples in ZF set theory. Hence the example generic axiomatic description paragraph adds the following association to the meaning of the specification.

$$(\_ \leftrightarrow \_) \mapsto \{(\mathit{set\ for}\ X, \mathit{set\ for}\ Y) \mapsto \mathit{value\ of\ powerset\ expression\ given\ that}\ X\ \mathit{and}\ Y,\\ \mathit{and\ so\ on\ for\ all\ combinations\ of\ sets\ for}\ X\ \mathit{and}\ Y\}$$

Syntactic transformation 12.2.3.4 moved the name of the generic operator to after the generic parameters. In constructing this association, the name had to be lifted back out again. This has sometimes been called the generic lifting operation.

Figure D.3 – Annotated parse tree of part of generic example



D.7 List arguments

An operator that forms an indexed-from-zero sequence can be introduced and defined as follows. The definition uses several toolkit notations.

function $(\langle^0 \; ,, \; \rangle^0)$

$$\langle^0 \; ,, \; \rangle^0[X] == \lambda\, s : seq\, X \bullet \{b : dom\;\; s;\;\; r : ran\;\; s \bullet b - 1 \mapsto r\}$$

The semantics of an application of this operator, for example $\langle^0 1, 2, 1\rangle^0$, are defined as follows.

D.7.1 Lexing and parsing

The application is recognised as being an instance of the `NofixApp` rule in the concrete syntax, the word $\langle^0$ being lexed as a `L` token and the word $\rangle^0$ being lexed as a `SR` token. Between those brackets, $1, 2, 1$ is recognised as an `ExpressionList`.

`L NUMERAL, NUMERAL, NUMERAL SR`	$\langle^0 1, 2, 1\rangle^0$

D.7.2 Syntactic transformation

The `ExpressionList` is transformed to an expression by rule 12.2.12.

`L {(NUMERAL, NUMERAL),`
`(NUMERAL, NUMERAL),`
`(NUMERAL, NUMERAL)} SR`
	$\langle^0\{(1, 1), (2, 2), (3, 1)\}\rangle^0$

The `NofixApp` is transformed to an application expression.

`NAME {(NUMERAL, NUMERAL),`
`(NUMERAL, NUMERAL),`
`(NUMERAL, NUMERAL)}`
	$\langle^0\bowtie\rangle^0\;\{(1, 1), (2, 2), (3, 1)\}$

The operator notation has now been eliminated, and so the semantic definition proceeds as usual for the remaining notation. All applications of operator notation are eliminated by such syntactic transformations.

D.8 Mutually recursive free types

The standard notation for free types is an extension of the traditional notation, to allow the specification of mutually recursive free types, such as the following example.

$$\begin{array}{l} exp ::= Node\langle\!\langle \mathbb{N}_1 \rangle\!\rangle \\ \qquad\quad \mid Cond\langle\!\langle pred \times exp \times exp \rangle\!\rangle \\ \& \\ pred ::= Compare\langle\!\langle exp \times exp \rangle\!\rangle \end{array}$$

This specifies a tiny language, in which an expression *exp* can be a conditional involving a predicate *pred*, and a *pred* compares expressions. A more realistic example would have more kinds of expressions and predicates, and maybe other auxiliary types perhaps in mutual recursion with these two, but this small example suffices here.

Like the previous examples, the source text for this one has to be taken through the phases of mark-up, lexing, parsing, syntactic transformation and type inference. (There are no applicable characterisations or instantiations.) The focus here is on the semantic transformation of free types. (Strictly, the Cartesian products should be syntactically transformed first, but keeping them makes the following more concise.)

D.8.1 Semantic transformation

Transforming the above free types paragraph by rule 12.2.3.5 generates the following Z notation. The semantic transformation rules are defined in terms of concrete notation for clarity, which should itself be subjected to further transformations, though that is not done here.

D.8.1.1 Type declarations

$[exp, pred]$

D.8.1.2 Membership constraints

$Node : \mathbb{P}(\mathbb{N}_1 \times exp)$
$Cond : \mathbb{P}((pred \times exp \times exp) \times exp)$
$Compare : \mathbb{P}((exp \times exp) \times exp)$

D.8.1.3 Total functionality constraints

$\forall u : \mathbb{N}_1 \bullet \exists_1 x : Node \bullet x.1 = u$
$\forall u : pred \times exp \times exp \bullet \exists_1 x : Cond \bullet x.1 = u$
$\forall u : exp \times exp \bullet \exists_1 x : Compare \bullet x.1 = u$

D.8.1.4 Injectivity constraints

$\forall u, v : nat_1 \mid Node\ u = Node\ v \bullet u = v$
$\forall u, v : pred \times exp \times exp \mid Cond\ u = Cond\ v \bullet u = v$
$\forall u, v : exp \times exp \mid Compare\ u = Compare\ v \bullet u = v$

D.8.1.5 Portmanteau disjointness constraint

There are no disjointness constraints from the *pred* type as it has only one injection and no element values.

$$
\begin{array}{l}
\forall b_1, b_2 : \mathbb{N} \bullet \\
\quad \forall w : exp \mid \\
\quad\quad (b_1 = 1 \land w \in \{x : Node \bullet x.2\} \lor \\
\quad\quad\quad b_1 = 2 \land w \in \{x : Cond \bullet x.2\}) \\
\quad\quad \land (B_2 = 1 \land w \in \{x : Node \bullet x.2\} \lor \\
\quad\quad\quad b_2 = 2 \land w \in \{x : Cond \bullet x.2\}) \bullet \\
\quad\quad\quad\quad b_1 = b_2
\end{array}
$$

D.8.1.6 Induction constraint

$$
\begin{array}{l}
\forall w_1 : \mathbb{P}\, exp;\ w_2 : \mathbb{P}\, pred \mid \\
\quad\quad (\forall y : (\mu\, exp == w_1;\ pred == w_2 \bullet \mathbb{N}_1) \bullet \\
\quad\quad\quad Node\ y \in w_1) \land \\
\quad\quad (\forall y : (\mu\, exp == w_1;\ pred == w_2 \bullet pred \times exp \times exp) \bullet \\
\quad\quad\quad Cond\ y \in w_1) \land \\
\quad\quad (\forall y : (\mu\, exp == w_1;\ pred == w_2 \bullet exp \times exp) \bullet \\
\quad\quad\quad Compare\ y \in w_2) \bullet \\
\quad w_1 = exp \land w_2 = pred
\end{array}
$$

D.9 Chained relations and implicit generic instantiation

The semantics of chained relations is defined to give a meaning to this example,

$$\{1\} \neq \varnothing \subseteq \{2, 3\}$$

in which $\varnothing$ refers to the generic definition of empty set and so is implicitly instantiated, whilst rejecting the following example as being not well-typed,

$$\{(1, 2)\} \neq \varnothing \subseteq \mathbb{A}$$

because the single $\varnothing$ expression in the second example needs to be instantiated differently for the two relations.

To demonstrate how this is done, the former example is taken through syntactic transformation and type inference, including the filling in of the implicit instantiations.

D.9.1 Syntactic transformation

The chaining is transformed by the first `InfixRel` rule in 12.2.10 to a conjunction of relations in which the duplicates of the common expression are constrained to be of the same type by giving them the same annotation.

$$\{1\} \neq (\varnothing \mathbin{⨾} \tau) \wedge (\varnothing \mathbin{⨾} \tau) \subseteq \{2, 3\}$$

The third `InfixRel` rule in 12.2.10 transforms these two relations to membership predicates.

$$(\{1\}, (\varnothing \mathbin{⨾} \tau)) \in (\_ \neq \_) \wedge ((\varnothing \mathbin{⨾} \tau), \{2, 3\}) \in (\_ \subseteq \_)$$

The two operator names are transformed by the second rule in 12.2.8.3.

$$(\{1\}, (\varnothing \mathbin{⨾} \tau)) \in \bowtie\!\neq\!\bowtie \wedge ((\varnothing \mathbin{⨾} \tau), \{2, 3\}) \in \bowtie\!\subseteq\!\bowtie$$

This is now a phrase of the annotated syntax.

D.9.2 Type inference

Type inference on this example generates the annotations illustrated in Figure D.4. (The tool used to draw that figure has no $\bowtie$ symbol, so ∞ is used instead.)

Those reference expressions that refer to generic definitions have to be transformed to generic instantiation expressions for their meaning to be determined. This is done by the instantiation rule (13.2.3.3). It determines the generic instantiations by comparison of the generic type with the inferred type. For example, the references to $\varnothing$ have been given the type annotation $\mathbb{P}(\mathtt{GIVEN}\ \mathbb{A})$, which is the instance of $[X]\, \mathbb{P}(\mathtt{GENTYPE}\ X)$ in which $\mathtt{GENTYPE}\ X$ is $\mathtt{GIVEN}\ \mathbb{A}$. The desired instantiating expression is the carrier set of that type, which is $\mathbb{A}$. It is generated by the instantiation rule, which effects the following transformation.

$$\varnothing \mathbin{⨾} [X]\, \mathbb{P}\, X, \mathbb{P}(\mathtt{GIVEN}\ \mathbb{A}) \implies \varnothing[\mathbb{A} \mathbin{⨾} \mathbb{P}(\mathtt{GIVEN}\ \mathbb{A})] \mathbin{⨾} \mathbb{P}(\mathtt{GIVEN}\ \mathbb{A})$$

Similarly, the reference to $\bowtie\!\neq\!\bowtie$ has type $\mathbb{P}(\mathbb{P}(\mathtt{GIVEN}\ \mathbb{A}) \times \mathbb{P}(\mathtt{GIVEN}\ \mathbb{A}))$ which is the instance of $[X]\, \mathbb{P}(\mathtt{GENTYPE}\ X \times \mathtt{GENTYPE}\ X)$ in which $\mathtt{GENTYPE}\ X$ is $\mathbb{P}(\mathtt{GIVEN}\ \mathbb{A})$. The instantiating expression is the carrier set of that type, which is $\mathbb{P}\, \mathbb{A}$.

Similarly again, the reference to $\bowtie\!\subseteq\!\bowtie$ has type $\mathbb{P}(\mathbb{P}(\mathtt{GIVEN}\ \mathbb{A}) \times \mathbb{P}(\mathtt{GIVEN}\ \mathbb{A}))$ which is the instance of $[X]\, \mathbb{P}(\mathbb{P}(\mathtt{GENTYPE}\ X) \times \mathbb{P}(\mathtt{GENTYPE}\ X))$ in which $\mathtt{GENTYPE}\ X$ is $\mathtt{GIVEN}\ \mathbb{A}$. The instantiating expression is the carrier set of that type, which is $\mathbb{A}$.

D.10 Logical inference rules

This document does not attempt to standardise any particular deductive system for Z. However, the soundness of potential logical inference rules can be shown relative to the sets of models defined by the semantics. Some examples are given here.

The predicate true can be used as an axiom. The proof of this is trivial: an axiom p is sound if and only if $[\![\ p\ ]\!]^{\mathcal{P}} = Model$ (as given by the definition of soundness in 5.2.3), and from the semantic relation for truth predicates (15.2.4.2), $[\![\ \text{true}\ ]\!]^{\mathcal{P}} = Model$.

The inference rule with premise $\neg\neg\, p$ and consequent p is sound if and only if

$$[\![\ \neg\neg\, p\ ]\!]^{\mathcal{P}} \subseteq [\![\ p\ ]\!]^{\mathcal{P}}$$

(again as given by the definition of soundness in 5.2.3). By two applications of the semantic relation for negation predicate (15.2.4.3), this becomes

$$Model \setminus (Model \setminus [\![\ p\ ]\!]^{\mathcal{P}}) \subseteq [\![\ p\ ]\!]^{\mathcal{P}}$$

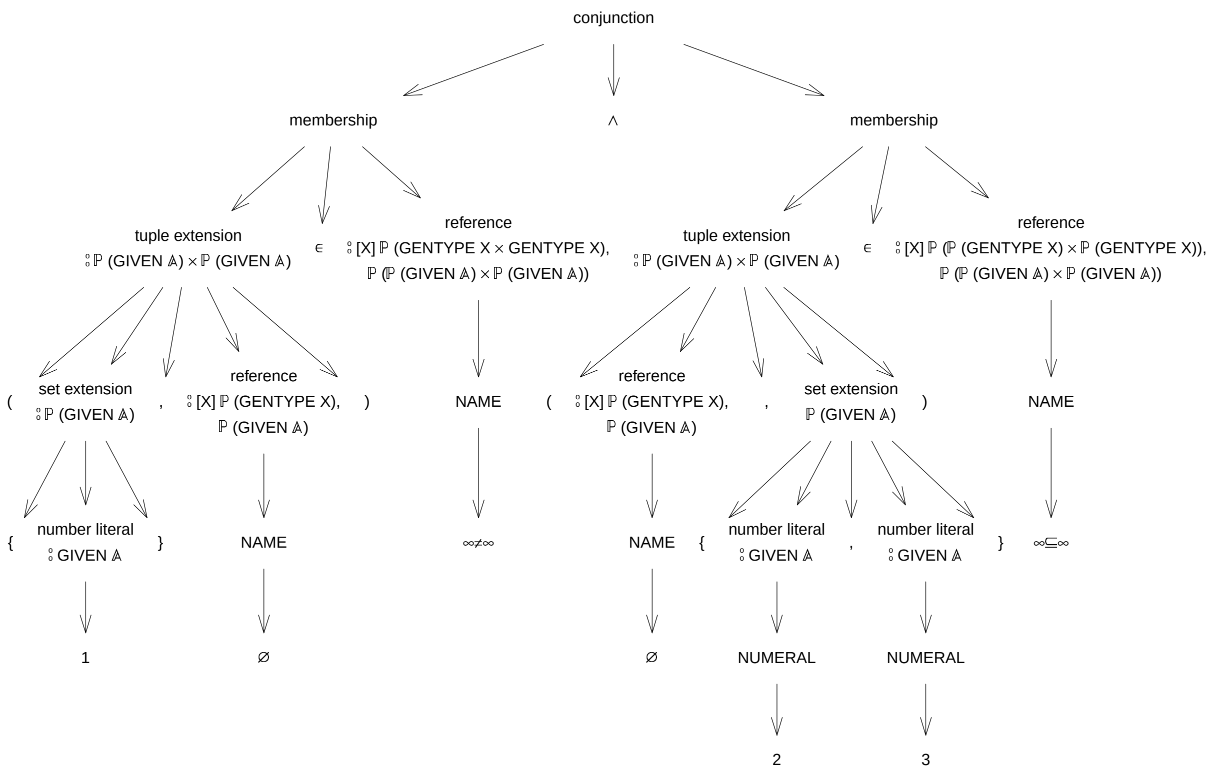
which by properties of set difference becomes

$$[\![\ p\ ]\!]^{\mathcal{P}} \subseteq [\![\ p\ ]\!]^{\mathcal{P}}$$

which is a property of set inclusion.

The transformation rules of clauses 12 and 14 inspire corresponding logical inference rules: any logical inference rule whose sole premise matches the left-hand side of a transformation rule and whose consequent is the corresponding instantiation of that transformation rule's right-hand side is sound.

Figure D.4 – Annotated parse tree of chained relation example



Annex E
(informative)

Conventions for state-based descriptions

E.1 Introduction

This annex records some of the conventions of notation that are often used when state-based descriptions of systems are written in Z. Conventions for identifying before and after states (x and x') and input and output variables ($i?$ and $o!$) are given.

E.2 States

When giving a model-based description of a system, the state of the system and the operations on the state are specified. Each operation is described as a relation between states. It is therefore necessary to distinguish between the values of state variables before the operation and their values afterwards. The convention in Z described here is to use dashes (primes) to make this distinction: if the state variables are x and y, then a predicate describing an operation is written using the variables x, y, x', y', where x and y denote the values before the operation, and x' and y' denote the values afterwards. (The predicate can also refer to any global constants.) For instance, if x and y are both integer variables, then an operation which incremented both variables could be specified as follows.

$$x' = x + 1 \land y' = y + 1$$

In order to use predicates like this to describe operations, all of the variables have to be in scope. If the state has been described in a schema S, then including S in the declaration part of the operation schema brings the state variables—x and y in the example above— into scope. The after-state variables are similarly introduced by including $S\,'$: this is a schema obtained from S by adding a dash to all the variables in the signature of S, and replacing every occurrence of such a variable in the predicate part of S by its dashed counterpart. Notice that the variables from the signature of S are the only ones which are dashed—global constants, types etc remain undashed. If S contains a variable which has already been decorated in some way, then an extra dash is added to the existing decoration.

Thus operations can be described in Z by a schema of the form

Op
S
$S\,'$
$\vdots$

Since the inclusion of undashed and dashed copies of the state schema is so common, an abbreviation is used:

$$\Delta S == [S;\ S\,']$$

The operation schema above now becomes

Op
ΔS
$\vdots$

It should be stressed that Δ is not an 'operator on schemas', merely a character in the schema name. One reason for this is that some authors like to include additional invariants in their Δ-schemas. For instance, if S contained an additional component z, but none of the operations ever changed z, then ΔS could be defined by

$$\Delta S == [S;\ S\,' \mid z' = z] \quad ,$$

thus making it unnecessary to include $z' = z$ in each operation description. If a name ΔS is referred to without a declaration of it having appeared previously, the reference is equivalent to $[S;\ S\,']$ as defined formally in 13.2.6.1.

It should be noted that strange results can occur if this implicit definition of ΔS is used on a schema S that contains variables which are not intended to be state components, perhaps inputs or outputs (see below). The sequence of strokes after a variable name might then become difficult to interpret.

There is one further piece of notation for describing state transitions: when enquiry operations are being described, it is often necessary to specify that the state should not change. For this the abbreviation ΞS is used. Unless it has been explicitly defined to mean something else, references to ΞS are equivalent to $[S;\ S\,' \mid \theta S = \theta S\,']$. Note that ΞS is not defined in terms of ΔS, in case ΔS has been given an explicit unconventional definition.

E.3 Inputs and outputs

For many systems, it is convenient to be able to describe operations not just in terms of relations between states, but with inputs and outputs as well. The input values of an operation are provided by 'the environment', and the outputs are returned to the environment.

In order to distinguish a variable intended as either an input or an output in an operation schema from a state-before variable (which has no decoration), an additional suffix is used: ? for input variables and ! for output variables. Thus *name*? denotes an input, and *result*! denotes an output.

E.4 Schema operators

The schema operators pre, $\fatsemi$ and $\gg$ make use of this decoration convention.

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